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Посібник призначено для використання студентами, що вивчають курс «Вища математика» в рамках підготовки фахівців економічного й інших напрямків підготовки бакалаврів. Посібник містить розділи: Лінійна алгебра. Матриці. Системи лінійних рівнянь. Векторна алгебра. Елементи аналітичної геометрії на площині й у просторі. Основи диференціального й інтегрального числення. Дослідження функцій. Диференціальні рівняння. Основи теорії імовірності й математичної статистики.

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PREFACE

This textbook is intended mainly for students who have already studied the basic Mathematics and need to study and practice using the methods of Differential and Integral Calculus. All the important concepts of Calculus are explained and there are exercises of each point to concentrate on those methods, which students need to use but which often cause difficulty. The mathematical language used is as simple as possible.

The textbook covers the topics to be studied:

1. LINEAR ALGEBRA. MATRICES. MATRIX OPERATION
2. LINES IN PLANE AND IN SPACE
3. CALCULUS. FUNCTIONS
4. THE DERIVATIVE.
5. INDEFINITE INTEGRAL. DEFINITE INTEGRAL. IMPROPER INTEGRAL
6. DIFFERENTIAL EQUATIONS
7. EQUATIONS OF MATHEMATICAL PHYSICS
8. ELEMENTS OF THE THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS

CONTENTS

Chapter 1. LINEAR ALGEBRA. MATRICES. MATRIX OPERATIONS

Definition (Def). *Matrix*. An array of numbers forming a rectangular table is called a matrix.

Def. The size or dimensions or order of a matrix are the number of rows and the number of columns it contains.

If there are m rows and n columns, the matrix is said to be m by *n*, which is written *m*n*.

Def. If *m=n* id est if a quantity of rows equals a quantity of columns, then the matrix is called square.

1.1. Matrix Operations

Def. If $A=(a_{ij})$ and $B=(b_{ij})$ are both *m*n* matrices, then their sum, $C=A+B$, is also *m*n* and its entries are given by the formula

$C_{ij}=a_{ij}+b_{ij}$

and their difference, $D=A-B$, is also *m*n* and its entries are given by the equation

$d_{ij}=a_{ij}-b_{ij}$.

Def. If $A=(a_{ij})$ is an *m*n* matrix and k is a scalar, then the scalar multiple $S=kA$ is also *m*n* and its entries are given by the formula $s_{ij}=ka_{ij}$.

Def. The transpose of an *m*n* matrix A is the *n*m* matrix A^T formed by making the rows of A the columns of A^T.

Def. Matrix multiplication. If A and B are matrices, then their product, AB, is defined only if the number of columns of A equals the number of rows of B. So, if the matrix A is *m*n*, then B must be *n*p* in order for the product AB to be defined. In this case, the size of the product matrix AB is *m*p*, and the (i,j) entry of AB is equal to the sum of products of entries of row i in A by corresponding entries of column j in B.

That is: (i,j) entry of $AB=(r_i \, in \, A)*(C_j \, in \, B)$.

Thus: $\underset{m*n}{A}\;\underset{n*p}{B}=\underset{m*p}{AB}$

Table 1

Basic definitions

English	Russian	Ukrainian
Matrix (matrices)	матрица	матриця
Array	построение, массив	побудова, масив
Rectangular	прямоугольный	прамокутний
Set	ряд	ряд
Row	строка	рядок
Column	столбец	стовпець
Restriction	ограничение	обмеження
Scalar	скаляр	скаляр
Row matrix (row vector)	матрица - строка	матриця - рядок
Column matrix (column vector)	матрица - столбец	матриця - стовпець
Vice versa	наоборот	навпаки
Transpose	транспонирование	транспонування
Id est (that is)	то есть	тобто

1.1.1. Determinants and their properties

Associated with each square matrix is an important number, called its determinant.

Def. *Determinant.* The determinant of the n-th order is a number or an algebraic expression corresponding to a square matrix with n^2 elements and calculating by the certain rules.

Method 1. Copy the first two columns of the determinant and place them to the right of it. Take the products formed by multiplying “down” and from their sum subtract the products formed by multiplying “up”.

Def. *Minor.* The minor M_{ij} associated with a_{ij} is obtained by blotting out of the determinant the row and column on which a_{ij} lies.

Method 2. The expansion along the column or row. The determinant equals the sum of the products of the entries of any line by their minors.

Theorem 1. The transpose determinant is equal to the original determinant.

Theorem 2. If two parallel lines – rows or columns of a determinant are interchanged, the determinant changes sign.

Theorem 3. If two parallel lines of a determinant are identical, then the determinant is O.

Theorem 4. If the entries in a line are all multiplied by a constant, then the determinant is multiplied by that constant.

Table 2

Basic definitions

English	***Russian***	***Ukrainian***
Concept	понятие	поняття
Determinant	определитель	визначник
To be of the form	иметь вид	мати вигляд
To evaluate	оценивать, находить	оцінювати, знаходити
Minor	минор	мінор
To obtain	получать, определять	отримувати, визначати
To blot out	вычёркивать	викреслювати
Expansion	разложение, расширение	розкладання, розширення
To interchange	менять местами	міняти місцями
To switch	поменять	поміняти

Task

If A is a 3*3 matrix whose determinant equals 5, what is the determinant of the matrix $2A$?

1.1.2. Identity matrices and inverses

Def. *A* square matrix, which has 1's along its main diagonal and O's elsewhere, is called an identity matrix and is denoted I.

Def. If both A and B are square matrices and $AB=I$ then A is called the inverse of B and B is called the inverse of A.

Def. A square matrix that has an inverse is said to be invertible.

1.2. Linear systems

Def. A linear system is a collection of a few linear equations for which we seek solutions (values of unknowns x_i) that satisfy all the equations of the system simultaneously.

Def. A system that has at least one solution is called consistent.

There are only 3 possibilities for the number of solutions:

1. There are no solutions. Such system is said to be inconsistent.
2. There is exactly one solution.
3. There are infinitely many solutions.

The graphs of the equations in the first case are parallel lines with no points in common. The graphs of the equations in the second case intersect in exactly one point. The graphs of the equations in the third system are lines that coincide.

Table 3

Basic definitions

English	***Russian***	***Ukrainian***
Identity matrix	единичная матрица	одинична матриця
Inverse	обратный	зворотний
Similarly	подобно, аналогично	подібно, аналогічно
Invertible	невырожденный	невироджений
Variable	переменный	змінний
Unknown	неизвестный	невідомий
Simultaneously	одновременно	одночасно
At least	по крайней мере	принаймні
Consistent	совместный	спільний
Infinitely	бесконечно	нескінченно
Graph	график	графік
Intersect	пересекать	перетинати
Coincide	совпадать	збігатися
distinct	различный	різний

Task

Two distinct solutions x_1 and x_2 can be found to the linear system $AX = B$. Which of the following is necessarily true?

a) $B = 0$; в) Ais invertible: c) $X_1 = -X_2$, d) there exists a solution x such that $x \neq x_1$, $x = x_2$.

1.2.1. Cramer's rule

It can be used for solving only a square linear systems.

If *A* is a square matrix, then linear system $AX = B$ has a unique solution for every *B* if and only if $\det A \neq 0$.

1.2.2. Gaussian elimination

1. Take the coefficients of the unknowns and form the coefficients matrix. Then attach the constants of the right-hand sides of the equations as an additional column, producing the augmented matrix.

2. Perform a series of elementary row operations to reduce (transform) the augmented matrix to echelon form.

A matrix is said to be in echelon form when it's upper triangular; any zero rows appear at the bottom of the matrix, and the first nonzero entry in any row appears to the right of the first nonzero entry in any higher row.

An elementary operations is one of the following:
a) multiplying a row by a nonzero coustant;
b) interchanging two rows;
c) adding a multiple of one row to another row.
3. Working from the bottom of the echelon matrix upward, evaluate the unknowns using backsubtitution.
To check the solutions plug it into all the original equations.

Table 4

Basic definitions

English	***Russian***	***Ukrainian***
unique	единственный	єдиний
to plug	подставить	підставити
elimination	исключение, устранение	виключення, усунення
to augment	увеличивать	збільшувати
upper triangular form	верхний треугольный вид	верхній трикутний вид
top row	верхняя строка	верхній рядок
bottom row	нижняя строка	нижній рядок
to yield	производить, получать	здійснювати, одержувати

Task

A driver wants to learn how many miles per gallon her car gets in the city and on the highway. On Monday she drove 30 miles in the city and 90 miles on the highway and used 6 gallons. During the 2-day period Tuesday and Wednesday, she drove75 miles in the city and 300 miles on the highway and used 17 galons. Thursday she drove 150 miles in the city and 210 miles on the highway and used 18 galons.
a) How much gasoline evaporates or leaks out of the tank per day?
b) How many miles per gallon does her car get in the city and on the highway?

1.3. The algebra of Vectors

Def. Two parallel directed line segments, P_1Q_1 and P_2Q_2, that have the same length and point in the same direction represent the same vectors.
Def. The vector, that has length 0 and no direction is called the zero vector.

Def. The length of the vector is called the magnitude and is denoted by $|\bar{a}|$. If the origin of a rectangular coordinate system is at the tail of $\bar{a}$, then the head of $\bar{a}$ has coordinates (x, y, z) in the space or (x, y) in the plane. The numbers x, and y and z are called the scalar components of $\bar{a}$ relative to the coordinate system.

Def. Any vectors of length unit is called a unit vector.

Def. The vectors $\bar{i} = (1, 0, 0)$, $\bar{j} = (0,1,0\ )$, $\bar{k} = (0,0,1\ )$ are called the basic unit vectors.

1.3.1. Algebraic operations on vectors

Def. The sum of two vectors $\bar{a}$ and $\bar{b}$ is defined as follows. Place the tail of $\bar{b}$ at the head of $\bar{a}$. Then the vector sum $\bar{a}+\bar{b}$ goes from the tail of $\bar{a}$ to the head of $\bar{b}$. Observe that $\bar{b}+\bar{a}=\bar{a}+\bar{b}$, since both sums lie on the diagonal of a parallelogram.

Def. Let $\bar{a}$ and $\bar{b}$ be vectors. The vector $\bar{v}$ such that $\bar{b}+\bar{v}$ equals $\bar{a}$ is called the difference of $\bar{a}$ and $\bar{b}$ and is denoted $\bar{a}-\bar{b}$. Thus $\bar{b}+\left(\bar{a}-\bar{b}\right)=\bar{a}$.

Def. The negative of the vector $\bar{a}$ is defined as the vector having the same magnitude as $\bar{a}$ but the opposite direction. It is denoted $-\bar{a}$. Observe that $\bar{a}+\left(-\bar{a}\right)=\bar{0}$, just as with scalars.

Def. The product of a scalar and a vector. If *k* is a scalar and $\bar{a}$ a vector, the product $k\bar{a}$ is the vector whose length is $|k|$ times the length of $\bar{a}$ and whose direction is the same as that of $\bar{a}$ if k is positive and opposite that of $\bar{a}$ if *k* is negative.

Theorem. For any vector $\bar{a}$ not equal to $\bar{o}$, the vector $\frac{\bar{a}}{|\bar{a}|}$ is the unit vector in the direction of $\bar{a}$.

Table 5

Basic definitions

English	***Russian***	***Ukrainian***
To point	указывать, показывать	указувати, показувати
Magnitude	величина, модуль (вектор)	величина, модуль (вектор)
Origin	начало	початок
Tail of a vector	начало вектора	початок вектора
Head of a vector	конец вектора	кінець вектора
Component	компонента	компонента
Unit vector	единичный вектор	одиничний вектор

To draw	чертить, строить	креслити, будувати
To magnify	увеличивать, растягивать	збільшувати, розтягувати

Task

1. Give an example of plane vectors $\bar{a}$ and $\bar{b}$ such that
   a) $|\bar{a}+\bar{b}| \neq |\bar{a}|+|\bar{b}|$,
   b) $|\bar{a}+\bar{b}| = |\bar{a}|+|\bar{b}|$.
2. Find the scalar components of $\bar{a}$ if
   a) $|\bar{a}|=10$, and $\bar{a}$ points to the north;
   b) $|\bar{a}|=6$, and $\bar{a}$ points to the southeast.
3. Let *a* and *b* be scalars, not both 0. Show that $\left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$ is a unit vector.
4. If $\bar{u}$ is a unit vector, what is the length of $-3\bar{u}$?
5. Find the unit vector that has the same direction as $\bar{i}+2\bar{j}+3\bar{k}$.

1.3.2. The dot product of two vectors

<u>Def.</u> Dot product. Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors. Their dot product is the number $|\bar{a}|\cdot|\bar{b}|\cdot\cos\theta$, where θ is the angle between $\bar{a}$ and $\bar{b}$. It is denoted $\bar{a}\cdot\bar{b}$. The dot product is a scalar and is also called the scalar product of $\bar{a}$ and $\bar{b}$.

If $\bar{a}$ is the force applied to an object and $\bar{b}$ is the line segment, then the dot product $\bar{a}\cdot\bar{b}$ defines the work accomplished by the force $\bar{a}$ in pulling the object along a straight line from the tail to the head of $\bar{b}$.

The angle between two vectors can be determined by the formula: $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}$.

<u>Def.</u> Let $\bar{a}$ and $\bar{b}$ be vectors. The projection of $\bar{a}$ on $\bar{b}$ is called the number $pr_b\bar{a} = |\bar{a}|\times\theta$, where θ is the angle between $\bar{a}$ and $\bar{b}$.

The direction of a vector in space involves three angles, two of which almost determine the third.

<u>Def.</u> Direction angles of a vector. Let $\bar{a}$ be a nonzero vectors. The angles between $\bar{a}$ and i, j, k are called the direction angels of a. They are denoted α, β, γ respectively. The numbers $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are the direction cosines of the vector $\bar{a}$.

$$\cos^2 a + \cos^2 \beta + \cos^2 \gamma = 1 .$$

In economics the dot product is used as an algebraic convenience.

Table 6

Basic definitions

English	***Russian***	***Ukrainian***
The dot product	скалярное произведение	скалярний добуток
Angel	угол	кут
Projection	проекция	проекція
Direction angle	направляющий угол	напрямний кут

Task

1. Compute $\bar{a}\square\bar{b}$:
   i. $\bar{a}$ has length 3, $\bar{b}$ has length 4 and the angle between $\bar{a}$ and $\bar{b}$ is $\pi/4$
   ii. $\bar{a} = 3\bar{i} - \bar{j} - 2\bar{k}$, $\bar{b} = \bar{i} + 5\bar{k}$.
   iii. $\bar{a} = \overline{MN}$, $\bar{b} = \ P$, where $M(4,-1,2)$, $P(2,-2,3)$, $Q(1,2,-7)$, $N(2,3,-4)$
2. Find the cosine of the angle between $\bar{i} + 6\bar{j} - \bar{k}$ and $4\bar{i} - \bar{j} - 2\bar{k}$.
3. Find the cosine of the angle between $\overline{AB}$ and CD if $A(0,-1,-2)$, $B(2,-1,3)$, $C(5,0,3)$, $D(-2,1,4)$.
4. Find the scalar components of $3\bar{i} - 2\bar{j}$ on $4\bar{j} + 3\bar{k}$.

1.3.3. The Cross Product. The Triple Scalar Product

<u>Def</u>. Let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{b} = b_1\bar{i} + b_2\bar{j} + b_3\bar{k}$.

The vector $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \bar{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \bar{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \bar{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$ is called the cross product of $\bar{a}$ and $\bar{b}$. It is denoted $\bar{a}\times\bar{b}$. The determinant of $\bar{a}\times\bar{b}$ is expanded along its first row.

Since the cross product of two vectors is a vector, the cross product is also called the vector product.

Note that $\bar{a}\times\bar{b}$ is a vector, while $\bar{a}\cdot\bar{b}$ is a scalar.

Theorem 1. The cross product $\bar{a}\times\bar{b}$ is a vector perpendicular to both $\bar{a}$ and $\bar{b}$.

So one of the most common uses of the cross product is in figuring out a vector normal to two given vectors.

Geometric Description of the Cross Product.

GD expresses the direction and magnitude of $\bar{a}\times\bar{b}$ in terms of those of $\bar{a}$ and $\bar{b}$.

To figure out the direction of the cross product, we use the right-hand rule: if the fingers of the right hand curl from $\bar{a}$ to $\bar{b}$ through an angle less than 180^0, then thumb points in the direction of $\bar{a}\times\bar{b}$.

Theorem. The magnitude of $\bar{a}\times\bar{b}$ is equal to the area of the parallelogram spanned by $\bar{a}$ and $\bar{b}$.

GD: $\bar{a}\times\bar{b}$ is that vector perpendicular to both $\bar{a}$ and $\bar{b}$, whose direction is obtained by the right-hand rule and whose length is the area of the parallelogram spanned by $\bar{a}$ and $\bar{b}$.

<u>**Def.**</u> The Triple Scalar Product. The scalar product of vectors $\left(\bar{a}\times\bar{b}\right)$ and $\bar{c}$ is called the triple scalar product. It is denoted $\bar{a}\bar{b}\bar{c}$.

Theorem. The absolute value of the triple scalar product is the volume of the parallelepiped formed by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Table 7

Basic definitions

English	***Russian***	***Ukrainian***
Right hand rule	правило “правой руки”	правило “правої руки”
To curl	завиваться	завиватися
Thumb	большой палец	великий палець
To span	соединять, покрывать, образовывать	з'єднувати, покривати, утворювати
Triple scalar product	смешанное произведение	мішаний добуток
Parallelogram	параллелограмм	паралелограм
Parallelepiped	параллелепипед	паралелепіпед

Task

1. Let $\bar{a}$ be a nonzero vector. If $\bar{a}\times\bar{b}=\bar{0}$ and $\bar{a}\cdot\bar{b}=0$, must $\bar{b}=\bar{0}$?
2. Show, that the points A(0, 1, 2), B(-2, 3, 0), C(1, 4, -2) and D(0, 9, 8) lie in the same plane.

Chapter 2. LINES IN PLANE AND IN SPACE

2.1. Lines in plane

Let $\bar{n} = A\bar{i} + B\bar{j}$ be a nonzero vector and (x_0, y_0) be a point in the xy plane. There is a unique line through (x_0, y_0) that is perpendicular to $\bar{n}$. Vector $\bar{n}$ is called a normal to the line.

Theorem 1. An equation of the line in the xy plane passing through (x_0, y_0) and perpendicular to the nonzero vector $\bar{n} = A\bar{i} + B\bar{j}$ is given by $A(x - x_0) + B(y - y_0) = 0$.

As theorem 1 shows, to find a vector perpendicular to a given line $Ax + By + C = 0$, form the vector $\bar{n} = A\bar{i} + B\bar{j}$. It will be perpendicular to the line. The constant term C plays no role in determining the direction of the line or of a vector perpendicular to it.

Theorem 2. The distance from the point $P_1(x_1, y_1)$ to the line L whose equation is $Ax + By + C = 0$ is $\dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

An equation of the line determined by two points: $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1}$.

2.1.1. Polar coordinates

Rectangular coordinates are only one of the way to describe points in the plane by pairs of numbers. Another system is called polar coordinates.

PC describe a point P as the interChapter of a circle and a ray from the center of that circle. They are defined as follows.

Select a point (pole) in the plane and a ray emanating from this point (polar axis). Measure positive angles θ counterclockwise from the polar axis and negative angles clockwise. Now let r be a number. To plot the point P that corresponds to the pair of numbers r and θ, proceed as follows:

If r is positive, P is the interChapter of the circle of radius r whose center is at the pole and the ray of angle θ, emanating from the pole. If r is θ, P is the pole, no matter what θ is.

If r is negative, P is at a distance $|r|$ from the pole on the ray directly opposite the ray of angle θ.

In each case P is denoted (r, θ).

2.1.2. The relations between polar and rectangular coordinates

$x = r\cos\theta$

$y = r\sin\theta;$

$r^2 = x^2 + y^2,\ tg\theta = \dfrac{y}{x}.$

Table 8

Basic definitions

English	***Russian***	***Ukrainian***
Polar	полярные координаты	полярні координати
Plane	плоскость	площина
Normal	нормаль	нормаль
Conversely	обратно	обернено
Inspection	осмотр	огляд
Right triangle	прямоугольный треугольник	прямокутний трикутник
Origin	начало (системы координат)	початок (системи координат)
Ray	луч	промінь
To emanate	исходить	виходити
Pole	полюс	полюс
To measure	измерять, откладывать, отмерять	вимірювати, відкладати, відміряти
Counterclockwise	в направлении против часовой стрелки	у напрямку проти годинникової стрілки
Clockwise	в направлении часовой стрелки	у напрямку годинникової стрілки
To go out	выходить	виходити

Task

1. Find the direction cosines of the line through the points $(4,-1)$ and $(-2,3)$.
2. Find the distance from the point $(-2,-3)$ to the line determined by the points $(0,4)$ and $(-3,7)$.

3. Give at least three pairs of polar coordinates (r,θ) for the point $\left(3,\frac{\pi}{4}\right)$.
4. Transform the equation into one in rectangular coordinates: $r=3$; $r=\sin\theta$.

2.1.3. Conic Chapters: ellipse, hyperbola, parabola

Def. The interChapter of a plane and the surface of a double cone is called a conic Chapter.

If the plane cuts off a bounded curve, that curve is called an ellipse. In particular, a circle is an ellipse.

If the plane is parallel to the edge of the double cone, the interChapter is called a parabola.

In the cases of the ellipse and the parabola, the plane generally meets just one of the two cones.

If the plane meets both parts of the cone and is not parallel to an edge, the interChapter is called a hyperbola. The hyperbola consists of two separate pieces.

For the sake of simplicity, we shall use a definition of the conic Chapters that depends only on the geometry of the plane.

Def. *Ellipse.* Let F and F' be points in the plane and let a be a fixed positive number such that $2a$ is greater than the distance between F and F'. A point P in the plane is on the ellipse determined by F, F' and $2a$ if and only if the sum of the distances from P to F and from P to F' equals $2a$. Points F and F' are the foci of the ellipse.

To construct an ellipse, place two tacks in a plane, tie a string of length $2a$ to them, and trace out a curve with a pencil held against the string, keeping the string taut by means of the pencil point.

The foci are at the tacks.

Def. The four points on the ellipse that are the furthest from or the nearest to the center are called vertices.

A circle does not have vertices.

Find the four vertices of the ellipse by checking where the curve intersects the x and y axes. Setting $y=0$ in equation, we obtain $x=a$ or $x=-a$; if we set $x=0$ in equation, we obtain $y=b$ or $y=-b$.

Thus the four vertices have coordinates $(a,0);(-a,0),(0,b) and (0,-b)$ Observe that the distance from F or F' to $(0,b)$ is a.

The right triangle with vertices F, $(0,b)$, and the origin, is a reminder of the fact that $b^2=a^2-c^2$.

Keep in mind that in above ellipse a is larger than b. The semimajor axis is said to have length a; the semiminor axis has length b.

Observe that we could interchange the roles of x and y and produce an ellipse with foci on y axis. In this case, y would have the larger denominator.

Table 9

Basic definitions

English	***Russian***	***Ukrainian***
Conic Chapter	коническое сечение	конічний перетин
Ellipse	эллипс	еліпс
Curve	кривая	крива
Bounded	замкнутый	замкнутий
Cone	конус	конус
Edge	край, ребро	край, ребро
Parabola	парабола	парабола
Hyperbola	гипербола	гіпербола
For the sake of simplicity	ради простоты	заради простоти
Focus	фокус	фокус
To tie	связывать, соединять	зв'язувати, з'єднувати
String	нить	нитка
To trance	чертить	креслити
Tout	туго натянутый	туго натягнутий
To get rid of	избавляться	позбуватися
Semimajor axis	большая полуось	більша піввісь
Semiminor axis	малая полуось	мала піввісь

Task

1. Find the equation of the ellipse with foci at $(0,3)$ and $(0,-3)$ such that the sum of the distances from a point on the ellipse to the two foci is 14.

2. Sketch the graph of the equation $\frac{x^2}{4}+\frac{y^2}{36}=1$ and its foci.

<u>Def.</u> <u>*Hyperbola.*</u> Let F and F' be points in the plane and let a be a fixed positive number such that $2a$ is less than the distance between F and F'. A point P in the plane is on the hyperbola determined by F, F' and $2a$ if and only if the difference between the distances from P to F and from P and F' equals $2a$ $(or -2a)$. Points F and F' are called the foci of the hyperbola.

A hyperbola consists of two separate curves.

Def. Asymptote. The lines $y = \frac{a}{b}x$ and $y = -\frac{a}{b}x$ are called asymptotes of the hyperbola.

It can be shown that the distance between points of hyperbola and points of its asymptotes approaches 0 when the points of the hyperbola move to infinity.

Def. *Parabola.* Let L be a line in the plane and let F be a point in the plane which is not on the line. A point P in the plane is on the parabola determined by F and L if and only if the distance from P to F equals the distance from P to the line L. Point F is the focus of the parabola; line L is its directrix. The point on the parabola nearest the directrix is called the vertex of the parabola.

2.1.4. Translation of axes and the graph of $Ax^2 + Cy^2 + Dx + Ey + F = 0$

The equation of any geometric object depends on where we choose to place the axes. Clearly, a wise choice of axes may yield a simpler way to choose convenient axes and uses the method to analyze equations.

A point P has coordinates (x, y) relative to a given choice of axes. Another pair of axes is chosen parallel to the first pair with its origin at the point (h, k). Call the second pair of axes the $x'y'$ axes.

Inspection of the figure shows that

$x' = x - h$, $y' = y - k$,

or equivalently,

$x = x' + h$, $y = y' + k$.

To transform the equation complete the square and use last formulas.

Table 10

Basic definitions

English	***Russian***	***Ukrainian***
Branch	ветвь	гилка
Upward	вверх	нагору
Downward	вниз	униз
To approach	приближаться	наближатися
Infinity	бесконечность	нескінченність
Asymptote	асимптота	асимптот
Directrix	директриса	директриса
To complete the square	выделить полный квадрат	виділити повний квадрат
Moreover	кроме того	крім того

Task

1. Using a suitable translation of axes, graph the equations relative to the xy axes:

a) $y=(x+1)^2$.

b) $y-2=2(x-1)^2$.

c) $y=2x^2-12x+20$.

d) $9x^2-4y^2-18x-27=0$.

2.1.5. Planes

A vector $\bar{n}$ is said to be perpendicular to a plane if $\bar{n}$ is perpendicular to every line situated in the plane.

We will consider the theorem, giving an algebraic condition that a point (x_0, y_0, z_0) must satisfy to be in a particular plane.

<u>**Theorem 3.**</u> An equation of the plane, passing through (x_0, y_0, z_0) and perpendicular to the nonzero vector $A\bar{i}+B\bar{j}+C\bar{k}$ is given by $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$.

<u>**Theorem 4.**</u> Let A, B, C and D be constant such that not all A, B and C are 0. Then the equation $Ax+By+Cz+D=0$ describes a plane. Moreover, the vector $A\bar{i}+B\bar{j}+C\bar{k}$ is perpendicular to this plane.

<u>**Theorem 5.**</u> The distance from the point (x_1, y_1, z_1) to the plane $Ax_1+By_1+Cz_1+D=0$ is $\frac{|Ax_1+By_1+Cz_1+D|}{\sqrt{A^2+B^2+C^2}}$.

An equation of the plane determined by three points.

Let we have three points $T_1(x_1, y_1, z_1)$, $T_2(x_2, y_2, z_2)$ and $T_3(x_3, y_3, z_3)$. If they don't lie on a single line, they determine a unique plane passing through them. Its equation is given by $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$.

<u>An angle between two planes.</u>

The angle between two planes is defined to be the angle between their normals, chosen so that the angle is at most $\frac{\pi}{2}$.

If the planes are perpendicular, the angle between them is $\frac{\pi}{2}$, hence $A_1A_2+B_1B_2+C_1C_2=0$. If the planes are parallel, their normals are parallel too, thus $\frac{A_1}{A_2}=\frac{B_1}{B_2}=\frac{C_1}{C_2}$.

Task

1. Find the distance from the point $(0,0,0)$ to the plane that passes through $(3,2,-1)$ and is perpendicular to vector $2\bar{i}+\bar{j}+\bar{k}$.

2. How far is the point $(2,3,-1)$ from the plane determined by the points $(1,1,1)$, $(-1,2,3)$ and $(3,-1,4)$?

2.2. Lines in space

Vectors provide a neat way to treat the geometry of lines in space.

Consider the line L through the point $P_0(x_0,y_0,z_0)$ and parallel to the vector $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$. A point $P(x,y,z)$ is on this line, if and only if the vector $\overline{P_0P}$ is a parallel to $\bar{a}$. One way to express that $\overline{P_0P}$ is parallel to $\bar{a}$ is to assert that there is a scalar t such that

$\overline{P_0P}=t\bar{a}$;

id est, $(x-x_0)\bar{i}+(y-y_0)\bar{j}+(z-z_0)\bar{k}=ta_1\bar{i}+ta_2\bar{j}+ta_3\bar{k}$.

Consequently, we have these <u>parametric equations</u> for the line through (x_0,y_0,z_0) parallel to $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$.

$$\begin{cases} x=x_0+ta_1 \\ y=y_0+ta_2 \\ z=z_0+ta_3. \end{cases}$$

Another way to express that $\overline{P_0P}$ is parallel to $\bar{a}$ is to use the condition when two vectors are parallel:

$$\frac{x-x_0}{a_1}=\frac{y-y_0}{a_2}=\frac{z-z_0}{a_3}.$$

If none of a_1,a_2,a_3 is 0, the equations are called symmetric equations of the line. These nonparametric equations describe the line as the interChapter of two planes

$$\frac{x-x_0}{a_1}=\frac{y-y_0}{a_2},\ \frac{y-y_0}{a_2}=\frac{z-z_0}{a_3}.$$

And this is the third way to determine the line in space.

<u>**Def.**</u> Direction numbers of the line. If vector $\bar{a}=a_1\bar{i}+a_2\bar{j}+a_3\bar{k}$ is parallel to the L then vector $\bar{a}$ is called direction vector of L.

Note that direction numbers and vector are not unique.

<u>Equation of the line through two points.</u>

Let we have two points $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$. In order to find the equation of the line through these points we can choose the vector $\overline{P_1P_2}$ as the direction vector of the line. Having substituted its coordinates into symmetric equations of the line we find $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$.

Table 11

Basic definitions

English	***Russian***	***Ukrainian***
Neat	стройный, лаконичный	стрункий, лаконічний
To assert	утверждать	затверджувати
Direction numbers	направляющие числа	напрямні числа
Direction vector	направляющий вектор	напрямний вектор
Parametric	параметрический	параметричний
Set	множество	множина

Task

1. Find the angle between the line through $(0,0,0)$ and $(1,1,1)$ and the plane through $(1,2,3)$, $(4,1,5)$, and $(2,0,6)$.

2. How far apart are the planes parallel to the plane $2x-5y+z+1=0$ that pass through the points $(1,2,3)$ and $(-1,0,4)$?

3. Where does the line through $(1,2,1)$ and $(3,1,1)$ meet the plane determined by the points $(2,-1,1), (5,2,3)$ and $(4,1,3)$?

4. Graph the plane and show its intercepts. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

2.2.1. Graph of equations

The set points (x,y,z) that satisfy some given equation in x,y and z is called the graph of that equation. For instance, the graph of the equation $Ax+By+Cz+D=0$, where not all of A,B and C are 0, is a plane.

<u>Def</u>. Cylinder. Let R be a set in a plane. The set formed by all lines that are perpendicular to the given plane and that meet R is called the cylinder determined by R.

Keep in mind that if an equation involves at most two of the letters x,y and z, its graph will be a cylinder in the space.

<u>Def</u>. The set of all points that are a fixed distance r from a given point (a,b,c) is a sphere of radius r and center (a,b,c).

To sketch this sphere, show the horizontal equator.

A point (x, y, z) is on this sphere when the distance between it and (a, b, c) is r.

Def. The graph of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, where a, b, c are positive constants, is called an ellipsoid.

In the special case when $a=b=c$ the equation becomes the equation of a sphere of radius a.

An ellipsoid meets the coordinate planes in ellipses.

To find where the ellipsoid meets a given axis, set the variables corresponding to the other two axes equal to 0.

The graph of $x^2+y^2+z^2=1$ is the sphere of radius 1 and center at the origin. By changing some of the plus signs to minus signs, we get new equations and graphs that are quite different from spheres.

If we make all three coefficients negative, the equation becomes $-x^2-y^2-z^2=1$, or $x^2+y^2+z^2=-1$. Since the left part of the equation is the sum of squares of real numbers, it is never negative; thus there are no points on that graph.

Next, the graphs of $x^2+y^2-z^2=1$ and $x^2-y^2-z^2=1$ turn out to be of interest and will introduce the “hyperboloid of one sheet” and “hyperboloid of two sheets”.

Def. For any positive numbers a, b, c the graph of $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is called a hyperboloid of two sheets.

Cross Chapters by planes parallel to the yz plane are ellipses, single points, or else empty. The cross Chapters by the xy and the xz planes are the hyperbolas.

Two minuses and one plus in any arrangement give a hyperboloid of two sheets.

Revolving the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ about the x axis produces a hyperboloid of two sheets; revolving it about the y axis a hyperboloid of one sheet.

Table 12

Basic definitions

English	***Russian***	***Ukrainian***
Cylinder	цилиндр	циліндр
To erect	сооружать, создавать	споруджувати, створювати
Sphere	сфера	сфера
Radius	радиус	радіус
Ellipsoid	эллипсоид	еліпсоїд
Various	различный	різний
Hyperboloid of one sheet	однополостный гиперболоид	одноплий гіперболоїд
Hyperboloid of two sheets	двуполостный гиперболоид	двуполий гіперболоїд
Revolution	поворот	поворот

Task

Sketch the given surfaces, showing any useful cross Chapter. Also describe its general appearance in words: include a description of cross Chapters and intercepts and tell whether it a surface of revolution.

a) $x^2+y^2+z^2+4y-2z-4=0$,

b) $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$,

c) $x^2+z^2=1$,

d) $x^2-\frac{y^2}{4}+z^2=1$,

e) $-x^2-y^2+z^2=1$,

f) $y+x^2=0$.

Chapter 3. CALCULUS. FUNCTIONS

<u>Def</u>. Let X and Y be sets. A function from X to Y is a rule or method for assigning to each element in X a unique element in Y.

A function may be given by a formula or a graph. It is often indicated by a table.

<u>Def</u>. Let X and Y be sets and let f be a function from X to Y. The set X is called the domain of the function. It $f(x) = y$, y is called the value of f at x. The set of all values of the function is called the range of the function.

The value $f(x)$ of a function f at x is also called the output, x is called the input or argument.

If $y = f(x)$, the symbol x is called the independent variable and the symbol y is called the dependent variable.

If both the inputs and outputs of a function are numbers, we shall call the function numerical or a real function of a real variable.

<u>Def</u>. <u>*Graph of a numerical function.*</u> Let f be a numerical function. The graph of f consists of those points (x, y) such that $y = f(x)$.

<u>Def</u>. <u>*Composition of functions*</u>. Let f and g be functions. Suppose that x is such that $g(x)$ is in the domain of f. Then the function that assigns to x the value $f(g(x))$ is called the composition of f and g. It is denoted $f \circ g$.

In other words to compute $f \circ g$, first apply g and then apply f to the result.

Certain functions behave nicely when composed with the function $-x$.

<u>Def</u>. <u>*Even function.*</u> A function f such that $f(-x) = f(x)$ is called an even function.

<u>Def</u>. <u>*Odd function*</u>. A function f such that $f(-x) = -f(x)$ is called an odd function.

Most functions are neither even no odd.

The graph of an even function is symmetric with respect to the y axis. The graph of an odd function is symmetric with respect to the origin.

<u>Def</u>. A function f that assigns distinct outputs to distinct inputs is called a one-to-one function.

The graph of a one-to-one function has the property that every horizontal line meets it in at most one point.

<u>Def</u>. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is an increasing function.

If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is a decreasing function.

These two types of functions are also called monotonic.

Def. Let $y = f(x)$ be a one-to-one function. The function g that assigns to each output of f the corresponding unique input is called the inverse of f.

Table 13

Basic definitions

English	***Russian***	***Ukrainian***
To indicate	показывать, представлять	показувати, представляти
Domain	область определения	область визначення
Range	область значений	область значень
Independent variable	независимая переменная	незалежна змінна
To compose	составлять	складати
Composition of functions	функция от функции, сложная функция	функція від функції, складна функція
Even function	четная функция	парна функція
Odd function	нечетная функция	непарна функція
One-to-one	однозначная функция	однозначна функція
Increasing function	возрастающая функция	зростаюча функція
Decreasing function	убывающая функция	убутна функція
Monotonic	монотонный	монотонний

Task

1. Describe the domain and range of the functions:

a) $f(x) = \dfrac{1}{\sqrt{x+1}}$; b) $f(x) = \dfrac{1}{1-x^2}$; c) $f(x) = \log_3(4+x^2)$.

2. For the given function evaluate and simplify the given expression

a) $f(x) = x^3$, $f(a+1) - f(a)$.

b) $f(x) = x + \dfrac{1}{x}$; $\dfrac{f(u) - f(v)}{u - v}$.

3. Express the given functions as compositions of two or more simpler functions.

a) $y=\sqrt{\frac{1}{1+2^x}}$; b) $y=\sin(3-\sqrt{x})$.

4. Let $f(x)=\sin x$. Is f one-to-one if the domain is taken to be:

a) the entire x axis?

b) the interval $[0,2\pi]$?

c) the interval $[0,\pi]$?

d) the interval $[-\frac{\pi}{2},\frac{\pi}{2}]$?

3.1. The limit of a function

The concept of a limit provides the foundation for both the derivative and the integral.

Consider a function f and a number a which may or may not be in the domain of f. In order to discuss the behavior of $f(x)$ for x near a, we must know that the domain of f contains numbers arbitrary close to a. Note how this assumptions is built into the following definitions.

Def. Limit of $f(x)$ at a. Let f be a function and a some fixed number. Assume that the domain of f contains open intervals (c,a) and (a,b). If there is a number L such that as x approaches a, either from the right or from the left, $f(x)$ approaches L, then L is called the limit of $f(x)$ as x approaches a.

3.1.1. One-sided limits

Def. Right-hand limit of $f(x)$ at a. Let f be a function and a some fixed number. Assume that the domain of f contains an open interval (a,b). If, as x approaches a from the right, $f(x)$ approaches a specific number L, then L is called the right-hand limit of $f(x)$ as x approaches a.

It is read "the limit of f of x as x approaches a from the right is L".

The left-hand limit is defined similarly. The only differences are that the domain of f must contain an open interval of the form (c,a) and $f(x)$ is examined as x approaches a from the left.

Note that if both the right-hand and the left-hand limits of f exist at a and are equal, then the limit of $f(x)$ as $x\to a$ exists. But if the right-hand and the left-hand limits are not equal, then the limit of $f(x)$ as $x\to a$ does not exist.

The tamest function are the constant function. A constant function assigns the same output to all inputs. If that fixed output is L, then $f(x)=L$ for all x. The graph of this function is a line parallel to the x axis.

Sometimes it is useful to know how $f(x)$ behaves when x is very large positive number or a negative number of large absolute value.

Rather than writing "as x gets arbitrary large through positive values, $f(x)$ approaches the number L", is customary to use the shorthand

It could be happen that as $x \to \infty$ a function $f(x)$ becomes and remains arbitrarily large and positive.

It is important, when reading the shorthand $\lim_{x \to \infty} f(x) = \infty$, to keep in mind that "$\infty$" is not a number.

Table 14

Basic definitions

English	***Russian***	***Ukrainian***
Limit	предел	границя
Derivative	производная	похідна
Integral	интеграл	інтеграл
Behavior	поведение	поведінка
Arbitrary close	сколь угодно близкие	як завгодно близькі
One-sided limit	односторонний предел	однобічна границя
Right-hand limit	правосторонний предел	правобічна границя
Left-hand limit	левосторонний предел	лівостороння границя
Tame	элементарный, простой	елементарний, простій

Task

Graph the function

$$f(x)=\begin{cases} -x & if \quad x<0 \\ 1 & if \quad x=0 \\ 2 & if \quad x>0 \end{cases}$$

and find

a) $\lim_{x \to -\infty} f(x)$;

b) $\lim_{x \to 0^-} f(x)$;

c) $\lim_{x \to 0^+} f(x)$;

d) $\lim_{x\to\infty} f(x)$;

e) $f(0)$.

3.1.2. Properties of limits

Theorem. Let f and g be two functions and assume that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist. Then

1. $\lim_{x\to a}(f(x)\pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$, id est the limit of the sum of two functions exists and equals the sum of the two given limits. This property extends to any finite sum of functions.

2. $\lim_{x\to a} f(x)\cdot g(x) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$. In particular, if $g(x)=k$, where k is any constant, $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$. Similarly this property extends to the product of any finite number of functions.

3. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

4. $\lim_{x\to a} f(x)^{g(x)} = (\lim_{x\to a} f(x))^{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} f(x) > 0$

3.1.3. Limits of a polynomial as $x\to\infty$ or $x\to-\infty$

It can be shown that if, as $x\to\infty$, $f(x)\to\infty$ and $g(x)=L>0$, then $\lim_{x\to\infty} f(x)\cdot g(x) = \infty$.

Def. A polynomial is a function of the form $a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, where $a_0, a_1, ..., a_n$ are fixed real numbers and n is a nonnegative integer. If a_n is not 0, n is the degree of the polynomial.

Let $f(x)$ be a polynomial of degree at least 1 and with the lead coefficient a_n positive.

Then $\lim_{x\to\infty} f(x) = \infty$.

It the degree of f is odd, then $\lim_{x\to-\infty} f(x) = -\infty$.

3.1.4. A contest between a large numerator and a large denominator

Let $f(x)$ be a polynomial and let ax^n be its term of highest degree. Let $g(x)$ be another polynomial and let bx^m be its term of highest degree.

Then $\lim_{x\to\pm\infty}\dfrac{f(x)}{g(x)} = \lim_{x\to\pm\infty}\dfrac{ax^n}{bx^m}$.

In short, when working with the limit of a quotient of two polynomials as $x \to \infty$ or as $x \to -\infty$, disregard all terms except the one of highest degree in each of the polynomials.

Table 15

Basic definitions

English	***Russian***	***Ukrainian***
To extend	распространять	поширювати
Finite	конечный	кінцевий
Polynomial	полином, многочлен	поліном, багаточлен
Degree	степень	ступінь
Lead coefficient	старший коэффициент	старший коефіцієнт
To disregard	пренебречь	зневажити

Let $P(x)$ be a polynomial of n, with lead term ax^n, $a>0$, and let $Q(x)$ be a polynomial of degree m, with lead term bx^m, $b>0$. Examine $\lim\limits_{x\to\infty}\frac{P(x)}{Q(x)}$ if

a) $m=n$, b) $m<n$, c) $m>n$.

1. Given that $\lim\limits_{x\to\infty} f(x)=\infty$ and $\lim\limits_{x\to\infty} g(x)=\infty$, discuss

a) $\lim\limits_{x\to\infty}(f(x)+g(x))$.

b) $\lim\limits_{x\to\infty}(f(x)-g(x))$.

c) $\lim\limits_{x\to\infty} f(x)g(x)$.

d) $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)}$.

3.1.5. Computations of limits

The technique of factoring out a power of x applies more generally than just to polynomials.

It was assumed that

$$\lim_{x\to\infty} f(g(x)) = f(\lim_{x\to\infty} g(x)).$$

For the functions f commonly met in calculus this switch of the order of "lim" and "f" is justified.

In case $\infty-\infty$ it is not immediately clear how this difference behaves. It is necessary to use a little algebra and rationalize the expression.

3.1.6. Asymptotes and their use in graphing

<u>Def</u>. If $\lim_{x\to\infty} f(x) = L$, where L is a real number, the graph of $y = f(x)$ gets arbitrary close to the horizontal line $y = L$ as x increases. The line $y = L$ is called <u>*a horizontal asymptote*</u> of the graph of f. An asymptote is defined similarly if $f(x) \to L$ as $x \to -\infty$.

<u>Def</u>. If $\lim_{x\to a^+} f(x) = \infty$ or if $\lim_{x\to a^-} f(x) = \infty$, the graph of $y = f(x)$ resembles the vertical line $x = a$ for x near a. The line $x = a$ is called <u>*a vertical asymptote*</u> of the graph of f. A similar definition holds if $\lim_{x\to a^+} f(x) = -\infty$ or $\lim_{x\to a^-} f(x) = -\infty$.

<u>Def</u>. The line $y = kx + b$ is a tilted asymptote of $f(x)$ if the function $f(x)$ may be represented of the form

$$f(x) = kx + b + \alpha(x),$$

where $\lim_{x\to\infty} \alpha(x) = 0$.

<u>Theorem</u>. In order to the graph of the function $f(x)$ have a tilted asymptote, it is necessary and suffices to exist the limits.

$$\lim_{x\to\infty} \frac{f(x)}{x} = k \text{ and } \lim_{x\to\infty}(f(x) - kx) = b$$

or $\lim_{x\to-\infty} \frac{f(x)}{x} = k$ and $\lim_{x\to-\infty}(f(x) - kx) = b$.

Table 16

Basic definitions

English	*Russian*	*Ukrainian*
Commonly met	часто встречающийся	що часто зустрічається
Switch	перестановка	перестановка
To resemble	быть похожим	бути схожим
Tilt	наклон	нахил

Task

1. Examine the given limits:

a) $\lim_{x\to\infty}(\sqrt{x^2 + 100x} - \sqrt{x^2 + 50x})$.

b) $\lim_{x\to\infty} \frac{\sqrt{2x^2 + 4x}}{\sqrt{4x^2 - 2}}$.

c) $\lim_{x\to -1^+} \frac{1}{(x+1)^2}$.

d) $\lim\limits_{x\to 0^+} \dfrac{1}{2^x - 1}$.

2. Use asymptote to sketch the graphs of the functions:

a) $f(x) = \dfrac{1}{(x+1)^2}$.

b) $f(x) = \dfrac{1}{x^3 + x^2}$.

c) $y = \dfrac{x^2}{x^2+1}$.

3.1.7. The limit of $(\sin\theta)/\theta$ *as* θ *approaches* 0

So far we found limits by algebraic means, such as factoring, rationalizing, or canceling. But some of the most important limits in calculus cannot be found so easily. To reinforce the concept of a limit and also to prepare for the calculus of trigonometric functions, we shall determine $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta}$.

Since both the numerator and the denominator, approach 0, this is a challenging limit.

<u>Theorem 1</u>. Let $\sin\theta$ denote the sine of an angle of θ radians. Then $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.

<u>The Squeeze Principle</u>. If $g(x) \le f(x) \le h(x)$ and $\lim\limits_{x\to a} g(x) = L$ and $\lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} f(x) = L$.

<u>Theorem 2</u>. Let $\cos\theta$ denote the cosine of an angle of θ radians. Then

$\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta} = 0$.

This implies that when θ is small, $1-\cos\theta$ is much smaller than θ.

From a practical point of view these limits showed that if angles are measured in radians, then the sine of a small angle is “roughly” the angle itself, that is $\sin x \approx x$.

<u>Def</u>. If $\lim\limits_{x\to a} \alpha(x) = 0$, $\lim\limits_{x\to a} \beta(x) = 0$ and $\lim\limits_{x\to a} \dfrac{\alpha(x)}{\beta(x)} = 1$, then the functions $\alpha(x)$ and $\beta(x)$ are equivalent. It may be proved that the following functions are equivalent as $x \to 0$:

$\sin x \approx \tan x \approx \arcsin x \approx \arctan x \approx x$

3.1.8. Natural logarithms

Let's discuss the limits: $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ and $\lim_{x\to\infty}(1+\frac{1}{x})^{x}$.

a) As $x\to 0$, the base $1+x$ approaches 1 and the exponent $\frac{1}{x}$ approaches ∞. The base 1influences the exponential function to be 1. The exponent ∞ influences the exponential function to be large. Thus this is a challenging limit.

b) As $x\to\infty$, the base $1+\frac{1}{x}$ approaches $1$and the exponent x approaches ∞. So this is the same case.

It was proved that both the limits exist and are equal.

Their value is denoted by number e and it is approximately equal to $e\approx 2,718\ldots$

Thanks to its useful properties the number e was chosen as a base of a special type of logarithm. It is called **natural logarithm** and is denoted $\ln x$.

That is $\ln x=\log_e x$.

Table 17

Basic definitions

English	***Russian***	***Ukrainian***
To reinforce	усиливать, подкреплять	підсилювати, підкріплювати
To challenge	требовать (внимания)	вимагати (уваги)
Radian	радиан	радіан
Squeeze	сжатие	стиск
To imply	подразумевать	мати на увазі
Roughly	грубо, приблизительно	грубо, приблизно
Estimate	оценка	оцінка
Base	основание степени	основа степеня
Exponent	показатель степени	показник степеня
Logarithm	логарифм	логарифм

Task

Examine the limits:

a) $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$.

b) $\lim_{h\to 0} h\coth$.

c) $\lim_{x\to 0^+} \frac{1-\cos x}{x^3}$.

1. What is domain of the function

$f(x) = \frac{\sin x}{x}$?

Show that $f(x)$ is an even function.

Find $\lim_{x\to\infty} f(x)$.

2. Find the limits

a) $\lim_{x\to 0}(1-5x)^{\frac{3}{x}}$.

b) $\lim_{x\to\infty}(\frac{x+4}{x})^{-2x}$.

3.2. Continuous functions

Usually we expect the output of a function at the input a to be closely connected with the outputs of the function at inputs that are near a. The functions of interest in calculus usually behave in the expected way; they offer no spectacular gaps or jumps. The graphs of these functions consist of curves or lines, not wildly scattered points. The technical term for these functions is *"continuous"*.

Def. Continuity from the right at a number a. Assume that $f(x)$ is defined at a and in some open interval (a,b). Then the function f is continuous at a from the right if $\lim_{x\to a^+} f(x) = f(a)$.

This means that

1. $\lim_{x\to a^+} f(x)$ exists and
2. that limit is $f(a)$.

Def. Continuity at a number a. Assume that $f(x)$ is defined in some open interval (b,c) that contains the number a. Then the function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$. This means that

1. $\lim_{x\to a} f(x)$ exists and
2. that limit is $f(a)$.

Def. Continuous function. Let f be a function whose domain is the x axis or is made up of open intervals. Then f is a continuous function if it is continuous at each number a in its domain.

Only a slight modification of the definition is necessary to cover functions whose domain involve closed intervals. We will say that a function whose domain is the closed interval $[a,b]$ is continuous if it is continuous at each point in the open interval (a,b), continuous from the right at a, and continuous from the left at b.

If f and g are defined at least in an open interval that includes the number a and if f and g are continuous at a, then so are $f+g$, $f-g$, fg. Moreover, if $g(a)\neq 0$, f/g is also continuous at a.

Let f be a continuous function. If g is some other function for which $\lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x))$.

That is for continuous f, "f" and "lim" can be switched.

Table 18

Basic definitions

English	***Russian***	***Ukrainian***
Continuous	непрерывный	безперервний
Spectacular	эффектный	ефектний
Gap	разрыв	розрив
To scatter	разбрасывать	розкидати
To amount	равняться	рівнятися
Slight	незначительный	незначний

Task

1. Let $f(x)$ equal the least integer that is greater or equal to x. For instance, $f(3)=3$, $f(3{,}4)=4$, $f(3{,}8)=4$. This function is sometimes denoted $[x]$ and called the *"ceiling"* of x.

a) Graph f.

b) Does $\lim\limits_{x\to 4^-} f(x)$ exist? If so, what is it?

c) Does $\lim\limits_{x\to 4^+} f(x)$ exist? If so, what is it?

d) Does $\lim\limits_{x\to 4} f(x)$ exist? If so, what is it?

e) Is f continuous at 4?

f) Where is f continuous?

g) Where is f not continuous?

2. Let $f(x)=x^2$ for $x<1$and let $f(x)=2x$ for $x>1$.

a) Graph f.

b) Can $f(1)$ be defined in such a way that f is continuous throughout the x axis?

3.3. The Maximum-Value Theorem and the_Intermediate-Value Theorem

Continuous function have two properties of particular importance in calculus: the "maximum-value" property and the "intermediate-value" property.

The first theorem asserts that a function that is continuous throughout the closed interval $[a,b]$ takes on a largest value somewhere in the interval.

It also takes on a smallest value.

3.3.1. Maximum-Value and Minimum-Value Theorem

Let f be continuous throughout the closed interval $[a,b]$. Then there is at least one number in $[a,b]$ at which f takes on a maximum value.

That is, for some number c in $[a,b]$.

$f(c) \geq f(x)$ for all x in $[a,b]$.

Similarly, f takes on a minimum value somewhere in the interval.

To persuade yourself that this theorem is plausible, imagine sketching the graph of a continuous function. As your pencil moves along the graph from some point on the graph to some other point on the graph, it passes through a highest point and also through the lowest point.

The maximum value theorem guarantees that a maximum value exists, but it does not tell how to find it.

The maximum and the minimum values of a function are called its extreme values or extrema.

To apply the maximum-value theorem, we must know that the function is continuous and the interval is closed, that is contains its endpoints: It can be shown that if either of these assumptions is deleted, the conclusion may be wrong.

3.3.2. Intermediate-Value Theorem

Let f be continuous throughout the closed interval $[a,b]$. Let m be any number between $f(a)$ and $f(b)$. (That is, $f(a) \leq m \leq f(b)$ if $f(a) \leq f(b)$, or $f(a) \geq m \geq f(b)$ if $f(a) \geq f(b)$). Then there is at least one number c in $[a,b]$ such that $f(c) = m$.

In other words, the intermediate value theorem reads:

A continuous function defined on $[a,b]$ takes on all values between $f(a)$ and $f(b)$. It asserts that a horizontal line of height m must meet the graph of f at least once if m is between $f(a)$ and $f(b)$.

When you move a pencil along the graph of a continuous function from one height to another, the pencil passes through all intermediate heights.

1. If a continuous function defined on an interval is positive somewhere in the interval and negative somewhere in the interval, then it must be 0 at some number in that interval.

2. To show that two functions are equal at some number in an interval, show that their difference is 0 at some number in the interval.

Table 19

Basic definitions

English	***Russian***	***Ukrainian***
Intermediate	промежуточный, средний	проміжний, середній
Persuade	убеждать	переконувати
Extreme value, extrema	экстремум	екстремум
Endpoint	граничные точки	межові точки
To attain	достигать	досягати
To guarantee	гарантировать	гарантувати

Task

1. Does the function $\frac{x^2+x}{x^4+3x+7}$ have a maximum value and a minimum value for x in $[1,5]$?

2. Show that the equation $x^5+3x^4+x-2=0$ has at least one root in the interval $[0,1]$.

3. Use the intermediate value theorem to show that the equation $3x^3+11x^2-5x=2$ has a solution.

4. Let $f(x)=\frac{1}{x}$, $a=-1$, $b=1$, $m=0$. Is there at least one c in $[a,b]$ such that $f(c)=0$?

If so, find c, if not, does this imply that the intermediate-value theorem is sometimes false?

Chapter 4. THE DERIVATIVE

One of the most important concepts of calculus is the derivative. It has a great number of applications.

First of all we will consider a few problems which at first glance may seem unrelated. But a little arithmetic will quickly show that they are all just different versions of one mathematical idea.

Problem Slope. What is the slope of the tangent line to the graph of $y = x^2$ at the point $P(x_0, y_0)$?

The slope of nonvertical line equals the quotient $\frac{y_2 - y_1}{x_2 - x_1}$, where $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are any distinct points on the line.

By the tangent line to a curve at a point P on the curve shall be meant the line through P that has the "same direction" as the curve at P.

In this case we formed a difference quotient,

$\frac{difference\_of\_outputs}{difference\_of\_inputs}$, and

examined its limit as the change in the inputs was made smaller and smaller.

The whole procedure can be carried out for another problems, for example seeking

- the velocity of a particle moving on a line,
- the density,
- the growth rate,
- the rate of profit
- the rate of change of any function.

The underlying common theme of these problems is the important mathematical concept, the derivative of a numerical function.

Def. The derivative of a function at the number x. Let f be a function that is defined at least in some open interval that contains the number x. If $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists it is called the derivative of f at x and is denoted $f'(x)$. The function is said to be differentiable at x.

Def. Velocity and speed of a particle moving on a line. The velocity at time t of an object whose position on a line at time t is given by $f(t)$ is the derivative of f at time t. The speed of the particle is the absolute value of the velocity.

Def. Density of material. The density at x of material distributed along a line in such a way that the left-hand x centimeters have a mass of $f(x)$ grams is equal to the derivative of f at x.

4.1. The derivative and continuity. Antiderivatives

If f is differentiable at each number x in some interval, it is said to be differentiable throughout that interval.

Theorem. If f is differentiable at a, then it is continuous at a.

Def. If f and F are two functions and f is the derivative of F, then F is called an antiderivative of f.

Table 20

Basic definitions

English	***Russian***	***Ukrainian***
Slope	наклон	нахил
Tangent	касательная, тангенс	дотична, тангенс
Secant	секущая, секанс	січна, секанс
Differentiable	дифференцируемый	диференційовний
Velocity, speed	скорость	швидкість
Particle	частица	частка
Density	плотность	щільність
To distribute	распределять	розподіляти
Rate	темп	темп
Antiderivative	первообразная	первісна
Change in the function	приращение функции	приріст функції

Task

1. Let $f(x)=x^3$.

a) Graph f.

b) On the graph show $x, x+\Delta x, \Delta x, f(x), f(x+\Delta x)$ and Δf for $x=2$ and $\Delta x=0{,}3$.

2. How many different antiderivatives does the function $f(x)$ have?

4.2. The Derivatives of the Sum, Difference, Product and Quotient

Consider methods for differentiating functions. Before developing the methods, it will be useful to find the derivative of any constant function.

Theorem 1. The derivative of a constant function is $0: c'=0$.

This theorem is no surprise: Since the graph of $f(x)=c$ is a horizontal line, it coincides with each of its tangent lines.

Also, if we think of x as time and $f(x)$ as the position of a particle, Theorem 1 implies that a stationary particle has zero velocity.

Theorem 2. If U and V are differentiable functions, then so is $U+V$. Its derivative is given by the formula $(U+V)'=U'+V'$.

Similarly, $(U-V)'=U'-V'$.

Theorem 2 extends to any finite number of differentiable functions.

The following theorem concerns the derivative of the product of two functions. The formula is more complicated than that for the derivative of the sum.

Theorem 3. If U and V are differentiable functions then so if UV. Its derivative is given by the formula $(UV)'=U'V+UV'$.

The theorem asserts that the derivative is the first function times the derivative of the second plus the second function times the derivative of the first.

By Theorem 3 $(cf)'=cf'$, where c is a constant, that is a constant factor can go past the derivative symbol.

Theorem 4. If u and v are differentiable functions, then so is u/v and $(\frac{U}{V})'=\frac{VU'-UV'}{V^2}$ where, V is not 0.

4.3. Composite Functions and the Chain Rule

If f and g are differentiable functions, is the composite function $f[g(x)]$ also differentiable? If so, what is its derivative? More concretely: If $y=f(U)$ and $U=g(x)$, then y is a function of x. How can we find dy/dx?

The Chain Rule. If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

That is, derivative of y with respect to x equals derivative of y with respect to U times derivative of U with respect to x.

The chain rule extends to a function built up as the composition of three or more functions.

Table 21

Basic definitions

English	*Russian*	*Ukrainian*
Differentiating	дифференцирование	диференціювання
Stationary	стационарный	стаціонарний
To go past the symbol	вынести за знак	винести за знак
Composite	сложный	складний
Chain	цепь	ланцюг
To allege	приписывать, утверждать голословно	приписувати, сверджувати голослівно

Task

1. Tell what is wrong with this alleged proof that $2=1$.

Observe that $x^2 = x.x = x + x + ... + x$ (x times).

Differentiation with respect to x yields the equation $2x = 1+1+...+1$ $(x1s)$. Thus $2x = x$. Setting $x=1$ shows that $2=1$.

2. Let f and g be differentiable functions. Shows that

a) $\frac{(fg)'}{fg} = \frac{f'}{f} + \frac{g'}{g}$.

b) $\frac{(f/g)'}{f/g} = \frac{f'}{f} - \frac{g'}{g}$.

3. Find an equation of the tangent line to the curve $y = x^3 - 2x^2$ at $(1,-1)$.

4.4. Applications of the derivative. Rolle's Theorem and the Mean-Value Theorem

Let f be a differentiable function defined at least on closed interval $[a,b]$. Because it is differentiable it is necessarily continuous. Hence the function f must take on a maximum value for some number c in $[a,b]$. That is, for some number c in $[a,b]$ $f(c) \geq f(x)$ for all x in $[a,b]$. What can be said about $f'(c)$?

First, if c is neither a no b, that is c is in the open interval (a,b), it seems likely that α tangent to the graph at $(c, f(c))$ would be parallel to the x axis, in which case $f'(c)=0$.

If, instead, the maximum occurs at an endpoint of the interval, at a or at b, the derivative at such a point need not be 0.

4.5. Theorem of the Interior Extremum

Let f be α function defined at least on the open interval (a,b). If f takes on a extremum value at a number c in this interval and if $f'(c)$ exists, then $f'(c)=0$.

<u>Def</u>. A line segment joining two points on the graph of a function f is called a chord of f.

Assume that a certain differentiable function f has a chord parallel to the x axis. It seems reasonable that the graph will then have at least one horizontal tangent line.

4.6. Rolle's Theorem

Let f be a continuous function on the closed interval $[a,b]$ and have a derivative at all x in the open interval (a,b). If $f(a)=f(b)$, then there is at least one number c in (a,b) such that $f'(c)=0$.

Rolle's theorem asserts that if the graph of a function has a horizontal chord, then is has a tangent line parallel to that chord. The mean-value theorem is a generalization of Rolle's theorem, since it concerns any chord of f, not just horizontal chords. In geometric terms, the theorem asserts that if you draw a chord for the graph, then somewhere above or below that chord the graph has at least one tangent line parallel to the chord.

4.7. Mean-Value theorem

Let f be a continuous function on the closed interval $[a,b]$ and have a derivative at every x in the open interval (a,b).Then there is at least one number c in the open interval (a,b) such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

<u>Corollary 1</u>. If the derivative of a function is 0 throughout an interval then the function is constant throughout that interval.

<u>Corollary 2</u>. If two functions have the same derivatives throughout an interval, then they differ by a constant. That is, if $f'(x)=g'(x)$ for all x in an interval, then there is a constant c such that $f(x)=g(x)+c$.

<u>Corollary 3</u>. If f is continuous on $[a,b]$ and has a positive (negative) derivative on the open interval (a,b), than f is increasing (decreasing) on the interval $[a,b]$.

Table 22

Basic definitions

English	***Russian***	***Ukrainian***
To occur	иметь место, случаться, попадаться	мати місце, траплятися, попадатися
Interior	внутренний	внутрішній
Chord	хорда	хорда
Mean-value theorem	теорема о среднем значении	теорема про середнє значення
Generalization	обобщение	узагальнення
Corollary	заключение, следствие, вывод	висновок

Task

1. Consider the function $f(x) = x^2$ only for x in $[-1, 2]$.
   a) graph the function $f(x)$ for $x \in [-1, 2]$.
   b) what is the maximum value of $f(x)$ for x in the interval $[-1, 2]$?
   c) does $f'(x)$ exist at the maximum?
   d) does $f'(x)$ equal 0 at the maximum?
   e) does $f'(x)$ equal 0 at the minimum?

2. Consider the function $f(x) = \dfrac{1}{x^2}$
   a) graph $f(x) = \dfrac{1}{x^2}$ for x in $[-1,1]$.
   b) show that $f(-1) = f(1)$.
   c) Is there a number c in $(-1,1)$ such that $f'(c) = 0$?
   d) Why does this function not contradict Rolle's theorem?

3. Using Corollary 1 of the mean-value theorem show that $f(x) = \cos^2 3x + \sin^2 3x$ is a constant. Find the constant.

4.8. Using the derivatives and limits when graphing a function

We'll consider how to use the derivative and limits to help graph a function. Of particular interest will be this questions:

Where is the derivative equal 0?

Where is the derivative positive? Negative?

How does the function behave for $|x|$ large?

<u>Def.</u> Critical number and critical points. A number c at which $f'(c) = 0$

is called a critical number for the function f. The corresponding point $(c, f(c))$ on the graph of f is a critical point on that graph.

Def. Relative maximum (local maximum). The function f has a relative (local) maximum at the number c if there is an open interval (a,b) around c such that $f(c) \geq f(x)$ for all x in (a,b) that lie in the domain of f. A local or relative minimum is defined analogously.

Def. Global maximum. The function f has a global (absolute) maximum at the number c if $f(c) \geq f(x)$ for all x in the domain of f. A global minimum is defined analogously.

By the theorem of the interior extremum, there is a close relation between a local extremum and critical points for a differentiable function. If a local extremum occurs at a number c that lies within some open interval within the domain of f, then $f'(c) = 0$. This means that c is a critical number. However, a critical point need not be a local extremum.

To determine whether a function has a local extremum at c, it is important to know how the derivative behaves for inputs near c.

4.9. First–derivative test for local maximum at $x = c$

Let f be function and let c be number in its domain. Assume that numbers a and b exist such that $a < c < b$ and

1. f is continuous on the open interval (a,b).
2. f is differentiable on the open interval (a,b), except possibly at c.
3. $f'(x)$ is positive for all $x < c$ in the interval and is negative for all $x > c$ in the interval.

Then f has a local maximum at c.

A similar test, which "positive" and "negative" interchanged, holds for a local minimum.

4.10. Higher derivatives

If $y = f(t)$ denotes position on a line at time t, then the derivative $\frac{dy}{dt}$ equals the velocity, and the derivative of the derivative, that is $\frac{d}{dt}(\frac{dy}{dt})$ equals the acceleration.

Most functions f met in applications of calculus can be differentiated repeatedly.

Def. The derivatives $f^{(n)}(x)$ for $n \geq 2$ are called the higher derivatives of f and are equal to derivative of $(n-1)$ *th* derivative.

Table 23

Basic definitions

English	***Russian***	***Ukrainian***
Relative (local) extrema	относительный (локальный) екстремум	відносний (локальний) екстремум
Higher derivatives	производные высших порядков	похідні вищих порядків
Acceleration	ускорение	прискорення

Task

1. Find the critical numbers of the given function and use the first – derivative test to determine whether a local maximum, a local minimum, or neither occurs there.

a) $3x^4 + x^3$.

b) $\frac{x^2}{2} - \ln x$.

c) $(x-1)^4$.

d) $x^2 \cdot e^{2x}$.

2. Graph the given function, showing any intercepts, asymptotes, critical points, or local or global exterma $\frac{x^2+3}{x^2-4}$.

3. Find all functions $f(x)$ such that $f^{(3)}(x) = 0$ for all x.

4.10.1. Concavity and the Second Derivative

Assume that $f''(x)$ is positive for all x in the open interval (a,b). Since f'' is the derivative of f', it follows that f' is an increasing function throughout the interval (a,b). In other words, if x increases, the slope of the graph of $y = f(x)$ increases as we move from left to right on that part of the graph corresponding to the interval (a,b). The slope may increase from negative to positive values, or the slope may be positive throughout (a,b) and increasing, or the slope may be negative throughout (a,b) and increasing.

Def. Concave upward. A function f whose first derivative is increasing throughout the open interval (a,b) is called concave upward in that interval.

It can be proved that where a curve is concave upward it lies above its tangent lines and below its chords.

Def. Concave downward. A function f whose first derivative is decreasing throughout an open interval (a,b) is called concave downward.

Where a function is concave downward, it lies below its tangent lines and above its chords. The sense of concavity is a useful tool in sketching the graph of a function. Of special interest is the presence of a point on the graph where the sense of concavity changes. Such a point is called an inflection point.

Def. Inflection point and inflection number. Let f be a function and let α be a number. Assume that there are numbers b and c such that $b<a<c$ and

1. f is continuous on the open interval (b,c).

2. f is concave upward in the interval (b,a) and concave downward in the interval (a,c), or vice versa.

The point $(a,f(a))$ is called an inflection point or point of inflection. The number a is called an inflection number. Observe that if the second derivative changes sign at the number a, then a is an inflection number.

If the second derivative exists at an inflection point, it must be 0. But there can be an inflection point even if f'' is not defined there.

4.10.2. The Second Derivative and local Extrema

Let a be a critical number for the function f and assume that $f''(a)$ happens to be negative. If f'' is continuous in some open interval that contains a, then $f''(a)$ remains negative for a suitably small open interval that contains a. This means that the graph of f is concave downward near $(a,f(a))$, hence lies below its tangent lines. In particular, it lies below the horizontal tangent line at the critical point$(a,f(a))$. Thus the function has a relative maximum at the critical number a.

Theorem. Second – derivative test for relative maximum or minimum. Let f be a function such that $f'(x)$ is defined at least on some open interval containing the number a. Assume that $f''(a)$ is defined. If $f'(a)=0, f''(a)<0$ then f has a local maximum at a. Similarly, if $f'(a)=0$ and $f''(a)>0$, then f has a local minimum at a.

Table 24

Basic definitions

English	***Russian***	***Ukrainian***
Concavity	вогнутость	увігнутість
Concave upward	вогнутый вверх	увігнутий нагору
Concave downward	вогнутый вниз	увігнутий униз
Inflection	изменение (перегиб)	зміна (перегин)
Sense	смысл, значение	сенс, значення
Presence	присутствие	присутність
Extent	размер, протяженность	розмір, довжина

Task

1. Sketch the general appearance of the graph of the given function near $(1,1)$ on the basis of the information given assume that f, f', f'' are continuous.

a) $f(1)=1$, $f'(1)=0$, $f''(1)=1$;

b) $f(1)=1$, $f'(1)=0$, $f''(1)=-1$;

c) $f(1)=1$, $f'(1)=0$, $f''(1)=0$ (sketch four possibilities).

2. Graph the functions, showing any relative maxima, relative minima, and inflection points.

a) $3x^5 - 5x^4$;

b) $\frac{x^2}{2}+\frac{1}{x}$.

4.11. General Procedure for Graphing a Function

Table 25

General Procedure for Graphing a Function

	Calculations	Geometric Meaning
Domain	1. Find where $f(x)$ is defined	Find horizontal extent of graph.
Intercepts	2. Find $f(0)$ and the values of x for which $f(x)=0$	Find where graph crosses the axes.
Critical numbers	3. Find where $f'(x)=0$	Find where the tangent line is horizontal.
Increasing, decreasing	4.Compute $f(x)$ at all critical numbers	Data needed for critical points.
	5. Find the values of x for which $f'(x)$ is positive and those for which $f'(x)$ is negative.	Find where graph goes up and where it goes down as pencil moves to the right.
Tilted asymptotes	6. Find $\lim\limits_{x\to\infty}\frac{f(x)}{x}=K$ and $\lim\limits_{x\to\infty}(f(x)-kx)$	Find tilted asymptote $y=kx+b$
Horizontal asymptotes	7. Find $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$	Find horizontal asymptotes or general behavior when $\|x\|$ is large.
Vertical asymptotes	8. Find the values of a where $\lim\limits_{x\to a^+}f(x)$ or $\lim\limits_{x\to a^-}f(x)$ is infinite	Find vertical asymptotes.
Concavity and inflection points	9. Find the values of x for which $f''(x)$ is positive and those for which $f''(x)$ is negative. Note where it changes sign	Find where the graph is concave upward and where it is concave downward. Note inflection points.
	10. Sketch the graph showing intercepts, critical points, asymptotes, local and global maxima and minima, and inflection points.	

4.12. Implicit Differentiation

Sometimes a function $y = f(x)$ is given indirectly by an equation that relates x and y. It is said to describe the function implicitly.

It is possible to differentiate a function given implicitly without having to solve for the function and express it explicitly. An example will illustrate the method, which is simply to differentiate both side of the equation that defines the function implicitly. This procedure is called implicit differentiation.

The problem could also be solved by differentiating explicit function. But the algebra involved is more complicated.

4.13. The Differential

The applied sciences are greatly concerned with the errors that may occur in measurements. Let $y = f(x)$ be a differentiable function. Then by the definition of a derivative, $\Delta y/\Delta x$ is a good approximation of $f'(x)$ when Δx is small. But on the other hand when Δx is small, the derivative $f'(x)$ is a good estimate of $\Delta y/\Delta x$.

<u>Def</u>. Let $y = f(x)$ be a differentiable function. Then $f'(x)\Delta x$ is called the differential of f and is denoted df or dy:

$dy = f'(x) = \Delta x$.

The differential can also be viewed geometrically. A very short piece of the graph around a point P, of a differentiable function, looks straight and closely resembles a short segment of the tangent line to the graph at P.

Thus the differential $f'(x)\Delta x$ represents vertical change along the tangent line.

The differential can be used to estimate the value of a function at the input $x + \Delta x$ in terms of information at x.

How to use a differential to estimate an output of a function

To estimate $f(b)$

1. Find a number a near b at which $f(a)$ and $f'(a)$ are easy to calculate.
2. Find $\Delta x = b - a$, Δx may be positive or negative.
3. Compute $f(a) + f'(a)\Delta x$. This is an estimate of $f(b)$. In short $f(b) \approx f(a) + (b-a)f'(a)$.

Table 26

Basic definitions

English	*Russian*	*Ukrainian*
Implicit	неявный	неявний
Explicit	явный, определённый	явний, певний

Task

1. Find $dy/\ dx$ at the indicated values of x and y in two ways: explicitly (solving for y first) and implicitly.

a) $x^2y + xy^2 = 12$ at $(3,1)$

b) $x^2 - y^2 = 3$ at $(2,1)$

2. Calculate the differentials, expressing them in terms of x and dx.

a) $d(\frac{\cos 5x}{x})$.

b) $d\sqrt{1+x^2}$.

c) $d(\tan x^3)$.

3. Use differentials to estimate the given quantities.

a) $\sqrt{103}$.

b) $Sin 32\circ$ (warning: First translate into radians)

Chapter 5. INDEFINITE INTEGRAL. DEFINITE INTEGRAL IMPROPER INTEGRAL

5.1. Indefinite integral

5.1.1. The antiderivatives and the indefinite integral

Def. If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

If $f(x)$ is a continuous function, then its antiderivative exists.

Theorem. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$ on an interval $[a,b]$, then there is a constant C such that

$F(x)= G(x) + C$

Def. A set of all antiderivatives of $f(x)$ is called an indefinite integral and is denoted

$\int f(x)dx = F(x) + C$

where $f(x)$ is called the integrand.

The process of finding an antiderivative is called integrating.

Def. The graph of any antiderivative is called an integral curve.

Every formula for a derivative provides a corresponding formula for an antiderivative.

Theorem. If $\int f(x)dx = F(x) + C$, then $\int f(ax + b)dx = 1/a\, F(ax + b) + C$ for any constants a and b.

Theorem. If $\int f(x)dx = F(x) + C$, then $\int f(u)du = F(u) + C$.

Where $u= \varphi(x)$ is any differentiable function of x.

Table 27

Basic definitions

English	***Russian***	***Ukrainian***
Antiderivative	первообразная	первісна
Indefinite integral	неопределенный интеграл	невизначений інтеграл
Integrand	подинтегральная функция	підінтегральна функція
Integrating	интегрирование	інтегрування
Integral curve	интегральная кривая	інтегральна крива

Task

1. Find dy/dx if $y = \int \sin(x^2)dx$.

2*. Verify the equation by differentiation

$$\int x^2 \sin ax dx = \frac{2x}{a^2}\sin ax + \frac{2}{a^3}\cos ax - \frac{x^2}{a}\cos ax + C$$

3. Compute the antiderivatives:

a) $\int (1+e^x)^2 dx$;

b) $\int \frac{dx}{\sqrt{18-2x^2}}$;

c) $\int \frac{dx}{7x+5}$;

d) $\int \frac{e^x}{1+e^x} dx$.

5.1.2. The substitution method

The substitution technique changes the form of an integral to that of an easier integral. It is the most commonly used technique of integration.

A substitution is worth trying in two cases:

1. The integrand can be written in the form of a product of a special type: function of *u(x)* x derivative of *u(x)* for some function *u(x).*

2. The integrand becomes simpler when a part of it is denoted *u(x).*

In order to apply the substitution technique to find $\int f(x)dx$ look for a function $u = h(x)$ *such* that $f(x) = g(h(x))\, h'(x)$, for some function *g*, or more simply, $f(x)dx = g(u)du$.

Then find an antiderivative of *g* and replace *u* by *h(x)* in this antiderivative.

It is important to keep in mind that there is no simple routine method for antidifferentiation of elementary functions.

<u>Theorem</u>. The substitution method. Let *g(u)* be a continuous function and let *h(x)* be a differentiable function. Assume that *G(u)* is an antiderivative of *g(u).* Then *G(h(x))* is an antiderivative of *g(h(x))h'(x)*. That is, if $G(u)=\int g(u)du$, then $G(h(x))=\int g(h(x))h'(x)dx$.

5.1.3. Integration by parts

The formula for the derivative of a product is a basis for integration by parts.

<u>Theorem</u>. Integration by parts. If *U* and *V* are differentiable functions, then

$\int UdV = UV - \int VdU$.

The key to applying integration by parts is the labeling of *U* and *dV*. Usually three conditions can be met:

1. *V* can be found by integrating and should not be too messy.
2. *dU* should not be messier then *U*.
3. *VdU* should be easier than the original *UdV*.

Table 28

Basic definitions

English	***Russian***	***Ukrainian***
Substitution method	метод подстановки	метод підстановки
Change of variables	замена переменных	заміна змінних
Label	обозначение	позначення
Messy	вызывающий затруднения	викликаючий труднощі
Integration by parts	интегрирование по частям	інтегрування вроздріб

Task

1. Use appropriate substitutions to find the antiderivatives

a) $\int \frac{e^x}{1+e^{2x}}dx$; b) $\int \frac{\cos\sqrt{t+1}}{\sqrt{t+1}}dt$; c) $\int x\cos x^2 dx$.

2*. Jack (using the substitution $u = \cos\theta$) claims that $\int 2\cos\theta\sin\theta d\theta = -\cos^2\theta$, while Jill (using the substitution $u = \sin\theta$) claims that the answer is $\sin^2\theta$.

Who is right ?

3. Find:

a) $\int \ln(7x-1)dx$;

b) $\int (3x^2-3x)\sin 2x dx$.

5.1.4. Integration by certain rational function. Integration of rational. Functions by partial fractions

Any rational function can be written of the form $\frac{P(x)}{Q(x)}$, where *P(x)* and *Q(x)* are polynomials.

The algebraic technique known as partial fractions makes it possible to integrate any rational function.

The technique of partial fractions depends on the result from advanced algebra: every rational function can be expressed as a sum of a polynomial and constant multiples of the three types of functions.

Since any polynomial and each of the three types of rational fractions can be integrated, any rational function can be integrated.

To express *P(x)/Q(x)*, where *P(x)* and *Q(x)* are polynomials, as the sum of partial fractions, follow these steps:

Step 1. If the degree of *P(x)* is equal to or greater than the degree of

Q(x), devide *Q(x)* into *P(x)* to obtain a quotient and a remainder:

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$$

where the degree of *R(x)* is less than the degree of *Q(x).*

<u>Step 2</u>. If the degree of *P(x)* is less than the degree of *Q(x),* then express *Q(x)* as the product of polinomials of degree 1 and 2, where the second – degree factors are irreducible.

<u>Step 3</u>. If *px + q* appears exactly *n* times in the factorization of *Q(x),* form the *sum*:

$$6\ \frac{k_1}{px+q} + \frac{k_2}{(px+q)^2} + \ldots + \frac{k_n}{(px+q)^n}$$

where the constant k_1, k_2 , … , k_n are to be determined later.

<u>Step 4</u>. If $ax^2 + bx + c$ appears exactly *m* times in the factorization of *Q(x),* then form the *sum:*

$$\frac{c_1x+d_1}{ax^2+bx+c} + \frac{c_2x+d_2}{(ax^2+bx+c)^2} + \ldots + \frac{c_mx+d_m}{(ax^2+bx+c)^m}$$

where the constants $c_1, c_2, \ldots , c_m$ and $d_1, d_2, \ldots, d_m$ are to be determined later.

<u>Step 5</u>. Determine the appropriate coefficients, such that *P(x)/Q(x)* is equal to the sum of all the terms formed in steps 3 and 4 for all factors of *Q(x)* defined in step 2. That may be done by the following way, called equating coefficients. It depends on the fact that if two polinomials are equal for all *x*, than corresponding coefficients must be equal.

Table 29

Basic definitions

English	***Russian***	***Ukrainian***
Irreducible	несократимый,	нескоротний,
Recursive	рекурентный	рекурентний
Quotient	частное	частка
Remainder	остаток	остача
Factorization	разложение на множители	розкладання на множники
Appropriate	соответствующий	відповідний

Task

1. Compute the integral:

a) $\int \frac{x^2-3x-1}{x(x+1)^2}dx$; b) $\int \frac{2x^3+4}{x^3+2x}dx$; c) $\int \frac{x^3}{x^2-x-6}dx$.

2*. a) Write $x^4 + x^2 + 1$ as the product of irreducible polinomials of second degree.

b) Compute $\int \frac{dx}{x^4 + x^2 + 1}$.

5.1.5. Integration of trigonometric functions

How to integrate any rational function of sinθ and cosθ.

A polinomial in *x* and *y* is a sum of terms of the form ax^iy^j, where *i* and *j* are nonnegative integers and *a* is a real number.

The quotient of two such polinomials is called a rational function of *x* and *y* and is denoted *R(x,y)*. If , in *R(x,y), x* and *y* are replaced by $cos\theta$ and $sin\theta$, we obtain a rational function of $cos\theta$ and $sin\theta$.

The technique of a particular substitution reduces the integration by any rational function of $cos\theta$ and $sin\theta$ to the integration of a rational function of *U*.

Task

1. Find the integrals

   a) $\int \cot^3 x dx$;

   b) $\int (\sin\theta + 2\cos\theta)^2 dx$;

   c) $\int \frac{d\theta}{4\cos\theta + 3\sin\theta}$.

5.1.6. Integration of rational function of x and roots

First of all let's consider trigonometric substitutions that turn certain rational function of quantities that involve square roots into rational functions of sinθ and cosθ; these can be integrated by corresponding methods.

5.1.7. Trigonometric substitutions

A rational function of x and $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, or $\sqrt{x^2 - a^2}$ can be integrated by using a trigonometric substitution. If the integrand is a rational function of x and

Case 1. $\sqrt{a^2 - x^2}$; let x = asinθ (a>0, -π/2 ≤ θ ≤ π/2).

Case 2. $\sqrt{a^2 + x^2}$; let x = atanθ (a>0, -π/2 < θ < π/2).

Case 3. $\sqrt{x^2 - a^2}$; let x = asecθ (a>0, 0 ≤ θ ≤ π, θ ≠ π/2).

The important thing that the square root sign disappears.

5.1.8. The algebraic substitution

Let n be a positive integer. Any rational function of x and $\sqrt[n]{ax+b}$ can be transformed into a rational function of U by the substitution

U = $\sqrt[n]{ax+b}$

and thus can be integrated by partial fractions.
Evaluate the integrals:

$$\int \frac{x^2}{1+x^6}dx, \int \sin^5 2x dx, \int \frac{1}{3+\cos x}dx, \int x^2 \sin 5x dx, \int \frac{x^4}{x^4-1}dx, \int \frac{1}{2\sqrt{x}-\sqrt[4]{x}}, \int \frac{x^3}{x-3}dx$$

5.2. The Definite Integral

We introduce the definite integral by an area problem.

5.2.1. An Area Problem

Find the area of the region bounded by the curve $y = f(x)$, the x axis, and the vertical lines $x = a$ and $x = b$. And let $f(x) \geq 0,\ x \in [a,b]$.

First, the interval $[a,b]$ is partitioned into n smaller Chapters, all of equal length or not.

After the division into n Chapters is formed a number is selected in each Chapter at which to evaluate $f(x)$.

Then above each small interval draw the rectangle whose height is $f(c_i)$.

The next step is to evaluate the function $f(x)$ at each c_i and form the sum with n summands – areas of all small rectangles.

It can be shown that the sums used to approximate the area, mass, distance, or volume were all made the some way.

Def. The sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ is called the approximating sum for the function $f(x)$ in interval $[a,b]$.

It is called a Riemann sum.

The larger n is and the shorter the Chapters are, the closer we would expect these approximating sums to be the quantity we are trying to find.

Def. Mesh. The mesh of a partition is the length of the longest Chapter in the partition.

Def. If $f(x)$ is a function defined on $[a,b]$ and the sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ approaches a certain number as the mesh of partitions of $[a,b]$ shrinks toward 0, no matter how the sampling number c_i is chosen, that certain number is called the definite integral of $f(x)$ over $[a,b]$.

Area, distance, mass, volume, are just particular interpretations of the definite integral.

Theorem. Existence of the definite integral. Let f be a continuous function defined on $[a,b]$. Then the approximating sum $\sum_{i=1}^{n} f(c_i)\Delta x_i$ approaches a single number as the mesh of the partition of *[a,b]* approaches 0. Hence $\int_a^b f(x)dx$ exists.

Mean-Value Theorem for Definite Integrals. Let a and b be numbers, and let *f* be a continuous function defined for *x* between *a* and *b*. Then there is a number *c* between *a* and *b* such that

$$\int_a^b f(x)dx = f(c)(a-b)$$

Table 30

Basic definitions

English	***Russian***	***Ukrainian***
Definite integral	определенный интеграл	визначений інтеграл
Area	площадь	площа
To partition	расчленять, разделять	розчленовувати, розділяти
To select	выбирать	вибирати
Sample	образец	зразок
Height	высота	висота
Rectangle	прямоугольник	прямокутник
Summand	слагаемое	доданок
Approximating sum	интегральная сумма	інтегральна сума
Mesh	мера	міра

Task

1. True or false:
   a) Every elementary function has an elementary derivative.
   b) Every elementary function has an elementary antiderivative.

5.2.2. The fundamental theorems of calculus

There is an intimate connection between the definite integral and the derivative. This relationship provides a tool for computing definite integrals. It is expressed in the fundamental theorems of calculus.

First Fundamental Theorem of Calculus. If *f* is continuous on *[a,b]* and if *F* is an antiderivative of *f*, then $\int_a^b f(x)dx = F(b) - F(a)$.

Second Fundamental Theorem of Calculus. Let *f* be continuous on an open interval containing the interval [a,b]. Let $G(x) = \int_a^x f(t)dt$ for $a \le x \le b$.

Then *G* is differentiable on *[a,b]* and its derivative is *f*; that is, *G'(x) = f(x).*

<u>Corollary</u>. Let *f* be continuous on an interval *[a,b]*. Then *f* is the derivative of some function.

The **First Fundamental Theorem** is abbreviated by the letters **FTC**. It provides a tool for computing many definite integrals. If an antiderivative of *f* is elementary, then FTC is of use. But there are elementary functions, for instance, $sinx^2$, $\sqrt{1+x^4}$, which are not derivatives of elementary functions. On these cases, it may be necessary to estimate the definite integral by an approximating sum.

Although there are formulas for computing definite integrals, do not forget that a definite integral is a limit of sums, because:

1. In many applications in science the concept of the definite integral is more important than its use as a computational tool.

2. Many definite integrals cannot be evaluated by a formula. Some of the more important of these have been tabulated to several decimal places and published in handbooks of mathematical tables.

5.2.3. The substitution method in the definite integral

Let *f* be a continuous function on a interval *[a,b], U = h(x)* be a differentiable function on the same interval, and *g* be a continuous function such that *f(x)dx=g(u)du*; that is *f(x) = g(h(x))h'(x).*

Then $\int_a^b f(x)dx = \int_{h(a)}^{h(b)} g(u)du$

Table 31

Basic definitions

English	***Russian***	***Ukrainian***
Fundamental	основной	основний
Intimate	близкий, тесный	близький, тісний
Connection, relationship	связь	зв'язок
Tool	средства, метод	засоби, метод

Task

1. Use a substitution to evaluate the definite integral:

$$\int_1^e \frac{\ln(x)^3}{x}dx \quad \int_0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta \quad \int_0^1 \frac{2x^3+1}{x^2+2}dx \quad \int_1^2 \frac{e^x}{1+e^{2x}}dx$$

2. Evaluate the integrals by integration by parts:

$$\int_0^1 x^2 e^{2x} dx \quad \int_0^1 \tan^{-1} x dx \quad \int_1^4 x \ln 3x dx$$

5.2.4. Applications of the Definite Integral

It was shown that the area of a plane region bounded by the curve y = f(x), (f(x)≥0), the x axis, and the vertical lines x = a and x = b is equal to

Area = $\int_a^b f(x)dx$

Let f and g be two continuous functions such that f(x) ≥ g(x) for all x in the interval [a,b]. Let R be the region between the curve y = f(x) and the curve y = g(x) for x in [a,b].

Inspection of figure shows that the area of R is given by

Area = $\int_a^b [f(x) - g(x)]dx$

5.2.5. Computing volume by parallel cross Chapters

Let's consider a spatial region, a "solid", bounded by the given surface. Let *A(x)* be an area of the plane region inside the solid, that is, the cross Chapteral area.

To find the volume of some solid, follow these steps:

1. Choose an x axis.

2. For each plane perpendicular to that axis, find the area of the cross Chapter of the solid made by the plane. Call this area *A(x)*.

3. Determine the limits of integration, *a* and *b*, for the region.

4. Evaluate the definite integral $\int_a^b A(x)dx$.

Most of the effort is usually spent in finding the integrand *A(x)*.

5.2.6. Solid of revolution

A lot of solids can be viewed as the solid obtained by revolving the plane region about some axis. This is a special case of a "solid of revolution". Let *R* be a region in the plane and *L* a line in the plane. Assume that *L* does not meet *R* at all or that *L* meets *R* only at points of boundary. The solid formed by revolving *R* about *L* is called a solid of revolution. Let us see how to compute the volume of a solid of revolution when *R* is region under the curve *y = f(x)* and above the interval *[a,b]* and *L* is the x axis.

To find the volume, first find the area *A(x)* of a typical cross Chapter made by a plane perpendicular to the *x* axis corresponding to the coordinate *x*. This cross Chapter is a disk of radius *f(x)*. Thus $A(x) = \pi[f(x)]^2$.

Since the volume of a solid is the integral of its cross-Chapteral area, we conclude that $V = \int_a^b \pi[f(x)]^2 dx$.

Table 32

Basic definitions

English	***Russian***	***Ukrainian***
application	приложение	додаток
plane region	плоская фигура	плоска фігура
cross Chapter	поперечное сечение	поперечний переріз
spatial	пространственный	просторовий
solid	тело	тіло
solid of revolution	тело вращения	тіло обертання

Task

1. Sketch the finite regions bounded by the given curves. Then find their areas.

a) $y = x^2$, $y = 3x - 2$.

b) $y = 2x^2$, $y = x + 1$.

c)* $x = y^2$, $x = 3y - 2$.

2. A region R in the plane is revolved around the x axis to produce a solid of revolution. In each case:

a) draw the region,

b) draw the solid of revolution,

c) draw the typical cross Chapter,

d) set up a definite integral for the volume,

e) evaluate the integral.

3. R is bounded by $y = \sqrt{x}$, the x axis, $x = 1$, $x = 2$.

4. R is bounded by $y = x^2$ and $y = x^3$.

5.3. Improper Integrals

5.3.1. Improper Integrals: Interval of Integration Unbounded

Def. Convergent improper integral. Let *f* be continuous for $x \geq a$. If $\lim_{в\to\infty} \int_a^b f(x)dx$ exists, the function *f(x)* is said to have a convergent improper integral from a to ∞. The value of the limit is denoted by $\int_a^\infty f(x)dx$.

Def. Divergent improper integral. Let *f* be a continuous function. If

$\lim\limits_{b\to\infty}\int_a^b f(x)dx$ does not exist, the function f is said to have a divergent improper integral from *a* to ∞.

An improper integral $\int_a^\infty f(x)dx$ can be divergent without infinite.

The improper integral $\int_{-\infty}^b f(x)dx$ is defined similarly.

If $\lim\limits_{a\to-\infty}\int_a^b f(x)dx$ exists, the improper integral is said to be convergent. If it does not exist, then the improper integral is said to be divergent. To deal with improper integrals over the entire x axis, define $\int_{-\infty}^\infty f(x)dx$ to be the sum $\int_{-\infty}^0 f(x)dx+\int_0^\infty f(x)dx$ which will be called convergent if both of them are convergent.

Sometimes $\int_a^\infty f(x)dx$ can be shown to be convergent by comparing it to another improper integral $\int_a^\infty g(x)dx$.

<u>Theorem 1</u>. Comparison test for improper integrals.

Let *f(x)* and *g(x)* be continuous functions *for* $x \geq a$. Assume that $0 \leq f(x) \leq g(x)$ and that $\int_a^\infty g(x)dx$ is convergent. Then $\int_a^\infty f(x)dx$ is convergent and $\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$

<u>Theorem 2</u>. Assume that *f(x)* is continuous for $x \geq a$, and assume that $\int_a^\infty |f(x)|dx$ is convergent. Then $\int_a^\infty f(x)dx$ is convergent.

5.3.2. Improper Integrals: Integrand Unbounded

<u>Def</u>. Convergent and divergent improper integrals. Let *f* be continuous at every number in *[a,b]* except *a*. If $\lim\limits_{t\to a+0}\int_t^b f(x)dx$ exists, the function *f* is said to have a convergent improper integral from *a* to *b*. If limit does not exist, the function *f* is said to have a divergent improper integral from *a* to *b*. In a similar manner, if *f* is not defined at *b*, define $\int_a^b f(x)dx$ as $\lim\limits_{t\to b-0}\int_a^t f(x)dx$, if this limit exists.

Chapter 6. DIFFERENTIAL EQUATIONS

6.1. Separable differential equations

An equation that involves one or more of the derivatives of a function is called a differential equation.

A solution of a differential equation is any function that satisfies the equation. To solve a differential equation means to find all its solutions.

The **order** of a differential equation is the highest order of the derivatives that appear in it.

We examine a special and important type of first-order differential equation, called **separable**. After showing how to solve it, we will apply it to the study of natural growth and decay and to inhibited growth.

A separable differential equation is one that can be written in the form

$$\frac{dy}{dx}=\frac{f(x)}{g(y)} \tag{6.1}$$

where $f(x)$ and $g(y)$ are differentiable functions. Such an equation can be solved by separating the variables, that is, bringing all the *x's* to one side and all the *y's* to the other side to obtain the following equation in differentials:

$$g(y)\,dy = f(x)\,dx. \tag{6.2}$$

This is solved by integrating both sides:

$$\int g(y)dy=\int f(x)dx+C \tag{6.3}$$

Some examples will illustrate the technique.

EXAMPLE 1. Solve $\frac{dy}{dx}=\frac{2x}{3y} \quad (y>0)$.

SOLUTION. Separating the variables, we obtain

$3y\,dy = 2x\,dx$.

Thus $\int 3y dy=\int 2x dx+C$

or $$\frac{3y^2}{2}=x^2+C\,. \tag{6.4}$$

Equation (6.4) determines y as a function of x implicitly. Each choice of C produces a solution.

EXAMPLE 2. Solve the differential equation

$$\frac{dy}{dx} = \frac{2y}{x} \qquad (x, y > 0)$$

(6.5)

SOLUTION. At first glance the equation does not appear to be of the form in Eq. (6.1). However, it can be rewritten in the form

$$\frac{dy}{dx} = \frac{(1/x)}{(1/2y)},$$

so it has the form of a separable differential equation. Separation of the variables is not hard:

$$\frac{dy}{dx} = \frac{2y}{x}, \quad \frac{dy}{dx} = \frac{2y}{x}.$$

Hence $\int \frac{dy}{2y} = \int \frac{dx}{x} + C$ or

$$\frac{1}{2}\ln y = \ln x + C$$

(6.6)

(since *x*, *y* assumed > 0, $\ln|x| = \ln x, \quad \ln|y| = \ln y$).

In this case, let us solve for *y* explicitly:

$\ln y = 2\ln x + 2C$

$y = e^{2\ln x + 2C}$ definition of natural logarithm

$y = e^{2\ln x}e^{2C}$ basic law of exponents

$y = (e^{\ln x})^2 e^{2C}$ power of a power

$y = x^2 e^{2C}$.

Since e^{2C} is an arbitrary positive constant, call it ***k*.** Thus the most general solution of Eq. (6.5) is

$$y = kx^2$$

(6.7)

As a check on this solution, see if $y = kx^2$ satisfies Eq. (6.5):

$$2kx = \frac{2kx^2}{x}.$$

Yes, it checks.

The solution of a separable differential equation (in fact, any first-order differential equation) will generally involve one arbitrary constant. Each choice of that constant determines a specific function that satisfies the differential equation.

6.2. The Differential Equations of Natural Growth and Decay

The next example treats a differential equation that is important in the study of growth and decay. It arises in such diverse areas as biology, ecology, physics, chemistry, and economic forecasting.

EXAMPLE 3. Solve the differential equation

$$\frac{dy}{dx} = ky \qquad (y > 0), \tag{6.8}$$

where *k* is a nonzero constant.

SOLUTION. Separation of the variables yields

$$\frac{dy}{y} = k \cdot dx$$

$$\int \frac{dy}{y} = \int k \cdot dx + C$$

$$y = e^{kx+C}$$

$$y = e^{C} \cdot e^{kx}.$$

Denote the arbitrary positive constant e^C by the letter *A***.** Then

$$y = Ae^{kx} \tag{6.9}$$

The most general solution of $dy/dx = ky$ is $y = Ae^{kx}$.

6.3. Linear differential equations with constant coefficients

This Chapter treats a type of differential equation that many engineering and physics students may meet even before they take a D.E. course. It is intended to serve as a reference.

The differential equation $\frac{dy}{dx} = a \cdot y$, or equivalently,

$$\frac{dy}{dx} - a \cdot y = 0 \tag{6.10}$$

was solved ealier. Any solution has to be of the form $y = A \cdot e^{a \cdot x}$ for some constant *A*. This Chapter is concerned with generalizations of Eq. (6.10).

First, we consider differential equations of the form

$$\frac{dy}{dx} + a \cdot y = f(x), \tag{6.11}$$

where *a* is a real constant and $f(x)$ is some function of *x*. [Equation (6.10) is the special case where $f(x) = 0$]. Equation (6.11) is called a first-order linear differential equation with constant coefficients. Second, we consider the second-order equation

$$\frac{d^2y}{dx^2}+b\cdot\frac{dy}{dx}+c\cdot y=f(x) \tag{6.12}$$

where *b* and *c* are real constants. For some *b* and *c*, solving Eq. (6.12) may use complex numbers even though the solution will be a real function. An engineer or physicist will meet Eq. (6.12) in the form

$$L\cdot\frac{d^2q}{dt^2}+R\cdot\frac{dq}{dt}+\frac{q}{C}=V\cdot\sin wt$$

in the study of electric currents. Here *q* is a charge that varies with time, *dqldt* is current, $V\sin\omega t$ describes an applied voltage, *R* is resistance, *L* is inductance, and *C* is a constant describing the capacitor. They also meet Eq. (6.12) in the study of motion in the form

$$m\cdot\frac{d^2x}{dt^2}+b\cdot\frac{dx}{dt}+k\cdot x=F_0\sin wt\,.$$

Here *x* describes the location of a particle moving on a line, $F_0\sin wt$ is an applied force, $b\cdot\frac{dx}{dt}$ describes a damping effect, $k\cdot x$ describes the force of a spring, and *m* is the mass.

Imagine for the moment that you have found a particular solution y_p of Eq. (6.11) and a solution y_1 of the associated *homogeneous* equation obtained from Eq. (6.11) by replacing $f(x)$ by 0, (The homogeneous case)

$$\frac{dy}{dx}+a\cdot y=0 \tag{6.13}$$

A straightforward computation then shows that y_p+y_1 is a solution of Eq. (6.11), as follows:

$$\frac{d}{dx}\left(y_p+y_1\right)+a\cdot\left(y_p+y_1\right)=\frac{dy_p}{dx}+\frac{dy_1}{dx}+a\cdot y_p+a\cdot y_1=$$

$$=\left(\frac{dy_p}{dx}+a\cdot y_p\right)+\left(\frac{dy_1}{dx}+a\cdot y_1\right)=f(x)+0=f(x).$$

Now, the function $y=C\cdot e^{-ax}$ for any constant C, is a solution of Eq. (6.13). Thus, if y_p is a solution of Eq. (6.11), then so is $y_p+C\cdot e^{-ax}$. In fact, each solution of Eq. (6.11) must be of the form $y_p+C\cdot e^{-ax}$. To see why, assume that y_p and *y* both satisfy Eq. (6.11). Then

$$\frac{d}{dx}\left(y-y_p\right)+a\cdot\left(y-y_p\right)==\left(\frac{dy}{dx}+a\cdot y\right)+\left(\frac{dy_p}{dx}+a\cdot y_p\right)=f(x)-f(x)=0.$$

Thus $y-y_p$, being a solution of Eq. (6.13), must be of the form Ce^{-ax}

for some constant C. Thus $y = y_p = Ce^{-ax}$. These observations are summarized in the following theorem.

Theorem 1. Let y_p be a particular solution of the differential equation

$$\frac{dy}{dx} + a \cdot y = f(x).$$

Then the most general solution is $y = y_p + Ce^{-ax}$

EXAMPLE 1. Solve the differential equation $\frac{dy}{dx} + 3 \cdot y = 12$.

SOLUTION. One solution is the constant function $y = 4$. The most general solution is, therefore, $y = 4 + Ce^{-3x}$ for any constant C.

Once a particular solution y_p has been found, Theorem 1 provides the general solution. Example 2 illustrates one technique for finding y_p.

EXAMPLE 2. Find all solutions of the differential equation

$$\frac{dy}{dx} - y = \sin x. \qquad (6.14)$$

SOLUTION. Start by guessing what a solution might look like. First find one solution. Since $f(x) = \sin x$, let us see if there is a solution of the form $y_p = A\cos x + B\sin x$, for some constants A and B. Substitution in Eq. (6.14) yields

$$\frac{d}{dx}(A\cos x + B\sin x) - (A\cos x + B\sin x) = \sin x.$$

So we want

$$-A\sin x + B\cos x - A\cos x - B\sin x = \sin x$$

or simply,

$$(-A - B)\sin x + (B - A)\cos x = \sin x.$$

Choose A and B such that $-A - B = 1$ and $B - A = 0$. It follows that $-A - (A) = 1$ or $A = -\frac{1}{2}$. Consequently,

$$y_p = -\frac{1}{2}\cos x - \frac{1}{2}\sin x$$

is a solution of Eq.(6.14), as may be checked by substitution in Eq. (6.14).

The general solution of the homogeneous equation $\frac{dy}{dx} - y = 0$ is $y = Ce^x$, so the general solution of Eq. (6.14) is $y = -\frac{1}{2}\cos x - \frac{1}{2}\sin x + Ce^x$.

Example 2 uses the method of undetermined coefficients: Guess a general form of the solution and see if the unknown constants can be chosen properly to yield a solution of the differential equation**.**

Before turning to solutions of Eq. (6.12), consider the special case when *f(x)* is identically 0, the so-called homogeneous case.

Let us find all solutions of the homogeneous equation

$$\frac{d^2y}{d^2x}+b\frac{dy}{dx}+cy=0. \qquad (6.15)$$

If y_1 and y_2 are both solutions of Eq. (6.15), a straightforward computation shows that $C_1y_1+C_2y_2$ is also a solution of Eq. (6.15) for any choice of constants C_1 and C_2. [Since Eq. (6.15) involves the second derivative of *y***,** we expect the general solution for *y* to contain two arbitrary constants.]

EXAMPLE 3. Solve

$$\frac{d^2y}{d^2x}-3\frac{dy}{dx}+2y=0. \qquad (6.16)$$

SOLUTION. Recalling our experience with Eq. (6.10), we are tempted to look for a solution of the form e^{kx} for some constant *k*. Substitution of e^{kx} into Eq. (6.16) yields

$$\frac{d^2(e^{kx})}{d^2x}-3\frac{d(e^{kx})}{dx}+2(e^{kx})=0,$$

or

$$k^2e^{kx}-3ke^{kx}+2e^{kx}=0,$$

which is equivalent to

$$k^2-3k+2=0. \qquad (6.17)$$

By the quadratic formula, *k = 1* or *k = 2*. Thus $y_1=e^x$ and $y_2=e^{2x}$ are solutions of Eq. (6.16). Consequently,

$$y=C_1e^x+C_2e^{2x} \qquad (6.18)$$

is a solution of Eq. (6.16) for any choice of constants C_1, and C_2. (It can be proved that there are no other solutions.)

The most general solution of the differential equation

$$\frac{d^2y}{d^2x}-6\frac{dy}{dx}+9y=0 \qquad (6.19)$$

is of a different form. If we try $y=e^{kx}$, we obtain

$$k^2e^{kx}-6ke^{kx}+9e^{kx}=0$$

$$e^{kx}(k^2+6k+9)=0$$

$$(k+3)^2=0$$

$$k=-3$$

This gives only the solutions of the form $y=Ce^{-3x}$. However, a second-order equation should possess a solution containing *two* arbitrary constants.

Let us seek all solutions of the form $y = v(x)Ce^{-3x}$,

hoping to find some not of the form $y = Ce^{-3x}$.

Straightforward computations give

$$\frac{dy}{dx} = v(x)(-3e^{-3x}) + v'(x)e^{-3x} = -3v(x)e^{-3x} + v'(x)e^{-3x}$$ and

$$\frac{d^2y}{dx^2} = 9v(x)e^{-3x} - 6v'(x)e^{-3x} + v''(x)e^{-3x}.$$

Substituting into Eq. (3.19) yields

$$9v(x)e^{-3x} - 6v'(x)e^{-3x} + v''(x)e^{-3x} - 18v(x)e^{-3x} + 6v'(x)e^{-3x} + 9v(x)e^{-3x} = 0$$

which simplifies to

$$v''(x)Ce^{-3x} = 0,$$

hence to

$$v''(x) = 0.$$

Therefore, $v(x) = C_1 + C_2x$**,** and our general solution is

$$y = C_1e^{-3x} + C_2xe^{-3x},$$

for arbitrary constants C_1, and C_2.

The key to the nature of the solutions of Eq. (6.15) lies in the associated quadratic

Equation $$t^2 + bt + c = 0 \qquad (6.20)$$

The type of solution to Eq. (6.15) depends on the nature of the roots of Eq. (6.20). There are three cases: two distinct real roots, a repeated root (necessarily real), and two distinct complex roots. Each case will be described by a corresponding theorem.

Theorem 2. If $b^2 - 4c$ is positive, Eq. (6.20) has two distinct real roots, r_1 and r_2 .In this case, the general solution of Eq. (6.15) is

$$y = C_1e^{r_1x} + C_2e^{r_2x}. \qquad (6.21)$$

The proof that $y = C_1e^{r_1x} + C_2e^{r_2x}$ is a solution is left to the reader. Theorem 2 covers the differential equation (6.16).

EXAMPLE 4. Solve $\dfrac{d^2y}{d^2x} - 6\dfrac{dy}{dx} + 9y = 0$.

SOLUTION. In this case, $b^2 - 4c = 21$, which is positive. The roots of the associated quadratic equation are

$$r_1 = \frac{-5+\sqrt{21}}{2} \text{ and } r_1 = \frac{-5-\sqrt{21}}{2}.$$

The general solution of the differential equation is

$$y = C_1e^{\frac{-5+\sqrt{21}}{2}\cdot x} + C_2e^{\frac{-5+\sqrt{21}}{2}\cdot x}.$$

The next theorem concerns the special case when the associated quadratic equation $t^2 + b \cdot t + c = 0$ has a repeated root, ***r*****.**

Theorem 3. If $b^2 - 4c = 0$, eq. (6.20) has a repeated root r. In this case, the general solution of Eq. (6.15) is

$$y = C_1 e^{r \cdot x} + C_2 \cdot x \cdot e^{r \cdot x} = (C_1 + C_2 \cdot x) \cdot e^{r \cdot x}.$$

That $y = (C_1 + C_2 \cdot x) \cdot e^{r \cdot x}$ is a solution is left to the reader to check by substitution. Theorem 3 is illustrated by the solution of Eq. (6.19).

Theorem 4. If $b^2 - 4c$ is negative, Eq. (6.20) has two distinct complex roots $r_1 = p + i \cdot q$ and $r_1 = p - i \cdot q$. In this case, the general solution of Eq. (6.15) is

$$y = (C_1 \cos qx + C_2 \sin qx) \cdot e^{px}. \qquad (6.22)$$

EXAMPLE 5. Find the general solution of the differential equation of harmonic motion,

$$\frac{d^2 y}{d^2 x} = -k^2 y, \qquad (6.23)$$

where *k* is a constant.

SOLUTION. Rewrite Eq. (6.23) in the form

$$\frac{d^2 y}{d^2 x} + k^2 y = 0,$$

which has the associated quadratic equation $t^2 + k^2 = 0$. The roots of this equation are $0 + ki$ and $0 - ki$. By Theorem 4, the general solution of Eq. (6.23) is

$$y = C_1 \cos kx + C_2 \sin kx.$$

Equation (6.23) describes the motion of a mass bobbing at the end of a spring. The height of the mass at time *x* is *y*. Since the motion is oscillatory, it is plausible that it is described by a combination of $\cos kx$ and $\sin kx$. If y_p is any particular solution of

$$\frac{d^2 y}{d^2 x} + b\frac{dy}{dx} + cy = f(x), \qquad (6.24)$$

and *y** is a solution of the associated homogeneous equation (6.15), then $y_p = y*$ is a solution of Eq. (6.24), as may be checked by a straightforward calculation. Since we know how to find the general solution of Eq. (6.15), all that remains is to find a particular solution of Eq. (6.24). This can often be accomplished by a shrewd guess and the use of undetermined coefficients, as illustrated by the following example.

EXAMPLE 6. Solve the differential equation

$$\frac{d^2 y}{d^2 x} + \frac{dy}{dx} + 2y = 2x^2 + 5. \qquad (6.25)$$

Since $2x^2+5$ is a polynomial, let us seek a polynomial solution. If there is such a solution, it cannot have degree greater than 2, since the right-hand side of Eq. (6.25) has degree 2. So try $y = Ax^2+Bx+C$; hence $y' = 2Ax+B$ and $y'' = 2A$. Substitution in Eq. (6.25) gives

$$2A+\left(2Ax+B\right)+2\left(Ax^2+Bx+C\right)=2x^2+5,$$

or $$2Ax^2+\left(2A+2B\right)x+\left(2A+B+2C\right)=2x^2+5.$$

Comparing coefficients gives $2A = 2$, $2A + 2B = 0$, and $2A + B + 2C = 5$. Thus $A = 1$, $B = -1$, and $C = 2$.

Consequently, $y_p = x^2 - x + 2$ is a particular solution of Eq. (6.25).

Next, turn to solving the associated homogeneous equation

$$\frac{d^2y}{d^2x}+\frac{dy}{dx}+2y=0. \qquad (6.26)$$

Here b = I and c = 2, so $b^2-4c=-7$. The roots of the associated quadratic equation $t^2+t+2=0$ are

$$\frac{-1\pm\sqrt{7}}{2}=\frac{-1}{2}\pm\frac{\sqrt{7}}{2}i.$$

By Theorem 4, the general solution of Eq. (6.26) is

$$y* = C_1e^{-\frac{x}{2}}\cos\frac{\sqrt{7}}{2}x+C_2e^{-\frac{x}{2}}\sin\frac{\sqrt{7}}{2}x$$

Putting everything together, we obtain the general solution of Eq. (6.25)

$$y = x^2-x+2+C_1e^{-\frac{x}{2}}\cos\frac{\sqrt{7}}{2}x+C_2e^{-\frac{x}{2}}\sin\frac{\sqrt{7}}{2}x.$$

Guessing a particular solution of Eq. (6.24) depends on the form of *f(x).* This table describes the most common cases:

Form of $f(x)$	**Guess for** y_p
A polynomial	Another polynomial
e^{kx} (*k* not a root of associated quadratic equation)	Ae^{kx}
xe^{kx} (*k* not a root of the associated quadratic equation)	$(A+Bx)e^{kx}$
$e^{kx}\sin qx$ or $e^{kx}\cos qx$ ($k+qx$ not a root of the associated quadratic equation)	$Ae^{kx}\cos qx+Be^{x}\sin qx$

A complete handbook of mathematical tables includes several pages of specific solutions for a much wider variety of functions *f(x)* that appear on the right side of Eq (6.24).

Chapter 7. EQUATIONS OF MATHEMATICAL PHYSICS

7.1. Basic types of equations of mathematical physics

The basic equations of mathematical physics (for the case of functions of two independent variables) are the following second-order partial differential equations:

I. Wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad (7.1)$$

This equation is used in the study of processes of transversal vibrations of a string, the longitudinal vibrations of rods, electric oscillations in wires, the torsional oscillations of shafts, oscillations in gases and so forth. This equation is an equation of hyperbolic type.

II. Fourier equation for heat conduction

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}. \qquad (7.2)$$

This equation is used in the study of processes of the propagation of heat, the filtration of liquids and gases in a porous medium (for example, the filtration of oil and gas in subterranean sandstones), some problems in probability theory. This equation is the simplest of the class of equations of parabolic type.

III. Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad (7.3)$$

This equation is invoked in the study of problems dealing with electric and magnetic fields, stationary thermal state, problems in hydrodynamics, diffusion. This equation is the simplest in the class of equations of elliptic type.

In equations (7.1), (7.2), and (7.3), the unknown function u depends on two variables. Also considered are appropriate equations of functions with a larger number of variables. The wave equation in three independent variables is of the form

$$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

The heat-conduction equation in three independent variables is of the form

$$\frac{\partial u}{\partial t} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Laplace equation in three independent variables has the form

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

7.2. Deriving the equation of the vibrating string. Formulating the boundary-value problem

In mathematical physics a string is understood to be a flexible and elastic thread. The tensions that arise in a string at any instant of time are directed along a tangent to its profile. Let a string of length l be, at the initial instant, directed along a segment of the x-axis from 0 to l. Assume that the ends of the string are fixed at the points $x=0$ and $x=l$. If the string is deflected from its original position and then let loose; or if without deflecting the string we impart to its points a certain velocity at the initial time, or if we deflect the string and impart a velocity to its points, then the points of the string will perform certain motions; we say that the string is set into vibration. The problem is to determine the shape of the string at any instant of time and to determine the law of motion of every point of the string as a function of time.

Let us consider small deflections of the points of the string from the initial position. We may suppose that the motion of the points of the string is perpendicular to the x-axis and in a single plane.

The process of vibration of the string is inscribed by a single function $u(x,t)$. A point of the string with abscissa x has moved at time t. Since we consider small deflections of the string in the x,u plane, we shall assume that the length of an element of string is equal to its projection on the x -axis. We also assume that the tension of the string at all points is the same; we denote it by T^*.

Consider an element of the string. Let us find the external forces applied to the element MN (Fig.1).

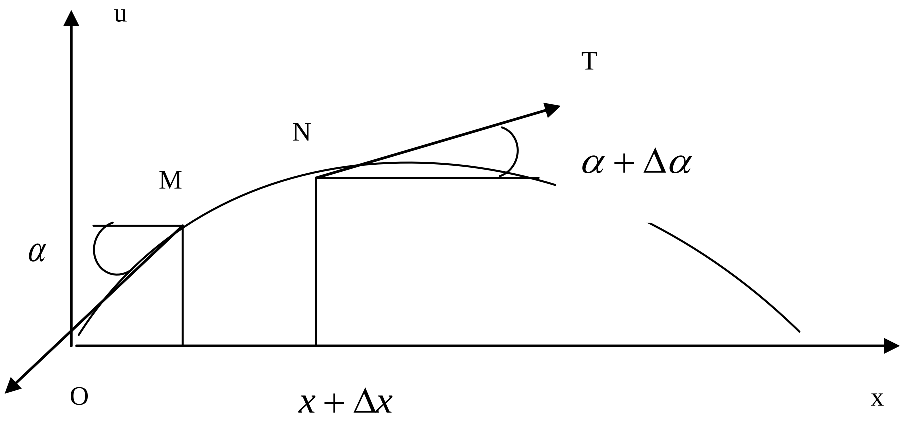


Fig. 1. The action of forces on the element of the string

$$T^* \sin(\alpha+\Delta\alpha) - T^* \sin\alpha \approx T^* \tan(\alpha+\Delta\alpha) - T^* \tan\alpha =$$
$$= T^* \left[\frac{\partial u(x+\Delta x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x} \right] = T^* \frac{\partial^2 u(x+\theta\cdot\Delta x,t)}{\partial x^2}\Delta x \approx T^* \frac{\partial^2 u(x,t)}{\partial x^2}\Delta x$$

(hear, we applied the Lagrange theorem for the expression in the square brackets).

In order to obtain the equation of motion, we must equate to the force of inertia the external forces applied to the element. Let ρ be the linear density of the string.

Then the mass of an x element of string, will be $\rho\Delta x$. The acceleration of the element is $\frac{\partial^2 u}{\partial t^2}$. By d'Alembert's principle we will have

$$\rho\cdot\Delta x\cdot\frac{\partial^2 u}{\partial t^2} = T^* \cdot \frac{\partial^2 u}{\partial x^2}\cdot\Delta x.$$

Canceling out Δx and denoting $\frac{T^*}{\rho} = a^2$, we get the equation of motion

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}. \tag{7.4}$$

This is the **wave equation**, the equation of the vibrating string. Equation (7.4) by itself is not sufficient for a complete definition of the motion of a string. The desired function $u(x,t)$ must also satisfy **boundary conditions** that indicate what occurs at the ends of the string and **initial conditions**, which describe the state of the string at the initial time $t=0$. The boundary and initial conditions are referred to collectively as **boundary-value conditions**.

Let the ends of the string at $x=0$ and $x=l$ be fixed. Then for any t the following equations must hold:

$$u(0,t)=0, \tag{7.5}$$

$$u(l,t)=0. \tag{7.6}$$

These equations are the boundary conditions for the problem.

$$u(x,0)=f(x), \tag{7.7}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0}=F(x). \tag{7.8}$$

Conditions (7.7) and (7.8) are the initial conditions.

7.3. Sollving of the equation of the vibrating String by the method of separation of variables (the Fourier method)

The method of separation of variables (or the Fourier method) is typical for solving of many problems in mathematical physics. Let it be required to find the solution of the equation (7.4) which satisfies the boundary-value conditions (7.5)-(7.6).

We shall seek a particular solution of equation (7.4) that satisfies the boundary conditions (7.5) and (7.6) , in the form of a product of two functions $X(x)$ and $T(t)$, of which the former is dependent only on x, and the letter, only on t:

$$u(x,t)=X(x)\cdot T(t). \tag{7.9}$$

Substituting into equation (7.1), we get

$$X(x)\cdot T''(t)=a^2X''(x)\cdot T(t),$$

and dividing the terms of the equation by $a^2X\cdot T$ we obtain

$$\frac{T''}{a^2T}=\frac{X''}{X}. \tag{7.10}$$

The left member of this equation is a function that does not depend on x, the right member is a function that does not depend on t. Equation (7.10) is possible only when the left and right members are not dependent either on x or on t, that is, are equal to a constant number. We denote it by $-\lambda$, where $\lambda<0$.It must be negative number to satisfy the boundary conditions (7.5) and (7.6). Thus,

$$\frac{T''}{a^2T}=\frac{X''}{X}=-\lambda.$$

From these equations we get two equations:

$$X''+\lambda\cdot X=0,$$

$$T''+a^2\lambda\cdot T=0.$$

The general solutions of these equations are

$$X(x)=A\cdot\cos\sqrt{\lambda}\cdot x+B\sin\sqrt{\lambda}\cdot x, \tag{7.11}$$

$$T(t) = C \cdot \cos\sqrt{\lambda} \cdot t + D\sin\sqrt{\lambda} \cdot t \qquad (7.12)$$

where A, B, C and D are arbitrary constants. Substituting the expressions $X(x)$ and $T(t)$ into (7.9), we get

$$u(x,t) = \left(A \cdot \cos\sqrt{\lambda} \cdot x + B\sin\sqrt{\lambda} \cdot x\right)\left(C \cdot \cos a\sqrt{\lambda} \cdot t + D\sin a\sqrt{\lambda} \cdot t\right).$$

Now choose the constants A and B so that the conditions (7.5) and (7.6) are satisfied. Since $T(t) \neq 0$, the function $X(x)$ must satisfy the conditions (7.5) and (7.6) that is, we must have

$$X(0) = 0, \; X(l) = 0.$$

Putting the values $x = 0$ and $x = l$ into (7.11), we obtain on the basis of (7.5) and (7.6)

$$0 = A \cdot 1 + B \cdot 0$$

$$0 = A \cdot \cos\sqrt{\lambda} \cdot l + B\sin\sqrt{\lambda} \cdot l$$

From the first equation we find $A = 0$. From the second it follows that

$$B\sin\sqrt{\lambda} \cdot l = 0.$$

$B \neq 0$, since otherwise we would have $X \equiv 0$ and $u \equiv 0$, which contradicts the hypothesis. Consequently, we must have

$$\sin\sqrt{\lambda} \cdot l = 0.$$

Whence

$$\sqrt{\lambda} = \frac{n\pi}{l} \quad (n = 1,2,...) \qquad (7.13)$$

(we do not take the value $n = 0$, since then we would have $X \equiv 0$ and $u \equiv 0$). And so we have

$$X = B \cdot \sin\frac{n\pi}{l}x. \qquad (7.14)$$

These values of λ are called **eigenvalues** of the given boundary-value problem. The functions $X(x)$ corresponding to them are called **eigenfunctions**.

It follows from (7.12)

$$T(t) = C\cos\frac{an\pi}{l}t + D\sin\frac{an\pi}{l}t \quad (n = 1,2,....). \qquad (7.15)$$

For each value of n, hence for every λ, we put the expressions (7.14) and (7.15) into (7.9) and obtain a solution of equation (7.4) that satisfies the boundary conditions (7.5) and (7.6). We denote this solution by $u_n(x,t)$:

$$u_n(x,t) = \sin\frac{n\pi}{l}x \cdot \left(C_n \cos\frac{an\pi}{l}t + D_n \sin\frac{an\pi}{l}t\right).$$

For each value of n we can take the constants C and D and thus write C_n and D_n (the constant B is included in C_n and D_n). Since equation (7.4) is linear and homogeneous , the sum of the solutions is also a solution, and therefore the function represented by the series

$$u(x,t)=\sum_{n=1}^{\infty}u_n(x,t) \quad \text{or}$$

$$u(x,t)=\sum_{n=1}^{\infty}\left(C_n\cos\frac{an\pi}{l}t+D_n\sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x \qquad (7.16)$$

will likewise be a solution of the differential equation (7.4), which will satisfy the boundary conditions (7.5) and (7.6). Series (7.16) will obviously be a solution of equation (7.4) only if the coefficients C_n and D_n are such that the series converges and that the series resulting from a double term-by-term differentiation with respect to x and to t converges as well.

This solution (7.16) should also satisfy the initial conditions (7.7) and (7.8). We may do this by choosing the constants C_n and D_n. Substituting $t=0$ into last equation, we get [see condition (7.7)]:

$$f(x)=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi}{l}x.$$

If the function $f(x)$ is such that in the interval $(0,l)$ it may be expanded in a Fourier series, the last equality will be fulfilled if we put

$$C_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi}{l}xdx. \qquad (7.17)$$

We then differentiate the terms of the function $u(x,t)$ with respect to t and substitute $t=0$. From condition (7.8) we get the equation

$$F(x)=\sum_{n=1}^{\infty}D_n\frac{an\pi}{l}\sin\frac{n\pi}{l}x.$$

We define the Fourier coefficients of this series

$$D_n=\frac{2}{an\pi}\int_0^l\varphi(x)\sin\frac{n\pi}{l}xdx. \qquad (7.18)$$

Thus, we have proved that the series (7.16), where the coefficients C_n and D_n are defined by formulas (7.17) and (7.18) [if it admits double termwise differentiation], is a function $u(x,t)$, which is the solution of equation (7.4) and satisfies the boundary and initial conditions (7.5)-(7.8).

Example. Determine the motion of the string under the boundary-value conditions (7.5)-(7.8). The initial deviation of the string is equal to zero but the initial rate of the motion is caused by hammer impact at the middle of the string. The functions $f(x)$ and $F(x)$ are determined by equalities

$$f(x)=0, \quad F(x)=\begin{cases}\dfrac{2h\cdot x}{l}\ \mathit{for} \quad 0\le x\le\dfrac{l}{2}, \\ \dfrac{2h\cdot(l-x)}{l}\ \mathit{for} \quad \dfrac{l}{2}\le x\le l.\end{cases}$$

The graph of the function $F(x)$ is shown on the figure 2. The ends of the string at $x=0$ and $x=l$ are fixed. Let us determined the Fourier coefficients. It follows from condition (7.7) $C_n = 0$. Let us find the Fourier coefficients D_n of the series

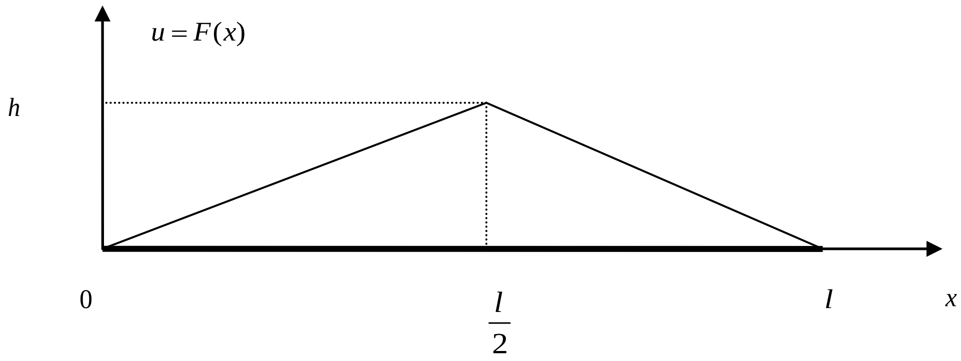


Fig. 2. The initial rate of the string

$$D_n = \frac{2}{an\pi}\int_0^l F(x)\sin\frac{n\pi}{l}xdx =$$

$$= \frac{2}{an\pi}\left(\int_0^{\frac{l}{2}}\frac{2h}{l}x\cdot\sin\frac{n\pi}{l}xdx + \int_{\frac{l}{2}}^{l}\frac{2h}{l}(l-x)\cdot\sin\frac{n\pi}{l}xdx\right).$$

Analysis of graphs of functions $F(x)$ and $\sin\frac{n\pi}{l}x$ that shown on the fig. 3 let simplify evaluation of coefficients D_n. Taking into account the symmetry of graphs we get the conclusion

$$\int_0^{\frac{l}{2}}\frac{2h}{l}x\cdot\sin\frac{n\pi}{l}xdx + \int_{\frac{l}{2}}^{l}\frac{2h}{l}(l-x)\cdot\sin\frac{n\pi}{l}xdx = 0,$$

if n is an even number, and

$$\int_0^{\frac{l}{2}}\frac{2h}{l}x\cdot\sin\frac{n\pi}{l}xdx + \int_{\frac{l}{2}}^{l}\frac{2h}{l}(l-x)\cdot\sin\frac{n\pi}{l}xdx = 2\cdot\int_0^{\frac{l}{2}}\frac{2h}{l}x\cdot\sin\frac{n\pi}{l}xdx,$$

if n is an odd number. Fourier coefficients D_n with even numbers disappear ($D_n = 0$, if n is an even number), but Fourier coefficients D_n are calculated by the formula

$$D_n = \frac{4}{an\pi}\int_0^{\frac{l}{2}}\frac{2h}{l}x\cdot\sin\frac{n\pi}{l}xdx,$$

if n is an odd number. Using the formula for integration by parts

$$\int_a^b u \cdot dv = u \cdot v\Big|_a^b - \int_a^b v \cdot du\,,$$

we get

$$D_n = \begin{pmatrix} u = \frac{2h}{l}x & du = \frac{2h}{l}dx \\ dv = \sin\frac{n\pi}{l}x \cdot dx & v = -\frac{l}{n\pi}\cos\frac{n\pi}{l}x \end{pmatrix} =$$

$$= \frac{4}{an\pi}\left(-\frac{2h}{l}x \cdot \frac{l}{n\pi}\cos\frac{n\pi}{l}x \Bigg|_0^{l/2} + \int_o^{\frac{l}{2}} \frac{l}{n\pi}\cos\frac{n\pi}{l}x \cdot \frac{2h}{l}dx \right) =$$

$$= \frac{4}{an\pi}\left(-\frac{h \cdot l}{n\pi}\cos\frac{n\pi}{2} + \frac{2h \cdot l}{n^2\pi^2}\sin\frac{n\pi \cdot l}{2} \right) = \frac{8h \cdot l}{a \cdot n^3\pi^3}\sin\frac{n\pi \cdot l}{2}\,.$$

Here n is an odd number. Let us make substitution $n = 2m - 1$ $(m = 1,2,3,...)$.

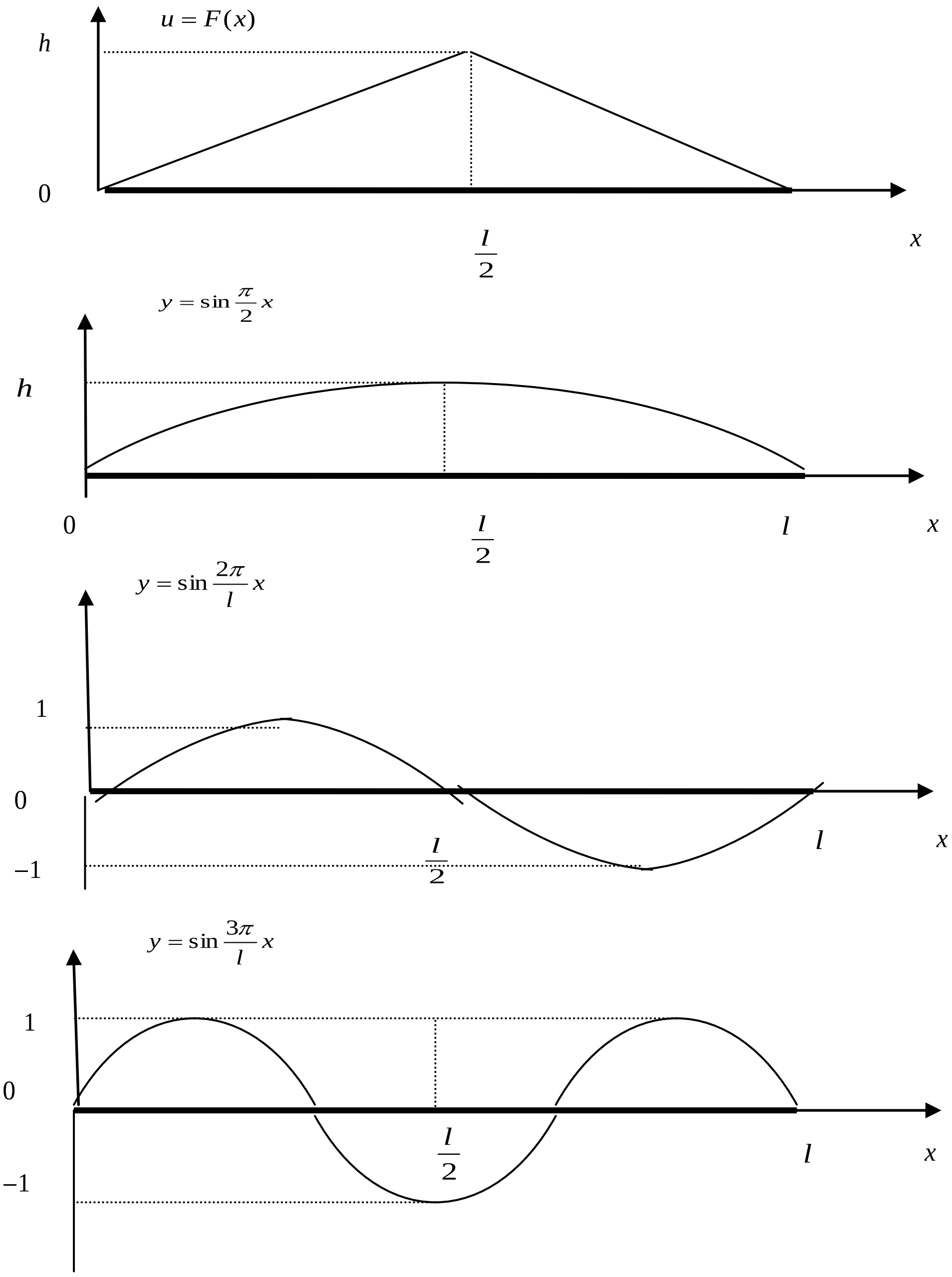


Fig.3. Graphs of the functions $F(x), \sin\frac{\pi}{2}x, \sin\frac{2\pi}{2}x, \sin\frac{3\pi}{2}x$

Then we obtain the answer for D_m:

$$D_m = \frac{8h \cdot l}{a \cdot (2m-1)^3 \pi^3} \cdot (-1)^{m-1}.$$

Taking into account equality (7.16) we get that the motion of the string is described by formula

$$u(x,t) = \frac{8h \cdot l}{a \cdot \pi^3} \cdot \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{(2m-1)^3} \sin\frac{a \cdot (2m-1) \cdot \pi \cdot t}{l} \sin\frac{(2m-1) \cdot \pi \cdot x}{l}. \qquad (7.19)$$

7.4. Solving of the equation of the vibrating endless. String by the running waves method (the D'alembert's method)

Now we will consider the motion of endless drown string. Let us imagine the ends of the string very far from the segment of it. We deflect this segment from its original position and impart a velocity to its points, then let loose. The string is set into vibration. We'll find a solution of the equation (7.1) satisfying the initial conditions (7.4) and (7.5) only. Such a problem is called the Cauchy's problem. We'll consider the D'alembert's method of solving the problem. It is called the **running waves method.** Let's prove the general solution of equation (7.1) has the form

$$u(x,t) = \varphi(x - a \cdot t) + \psi(x + a \cdot t). \qquad (7.20)$$

Here φ and ψ are arbitrary functions double differentiable with respect to x and t. Indeed

$$u'_x = \varphi'(x - at) + \psi'(x + at),$$
$$u''_{xx} = \varphi''(x - at) + \psi''(x + at),$$
$$u'_t = -a \cdot \varphi'(x - at) + a \cdot \psi'(x + at),$$
$$u''_{tt} = a^2 \cdot \varphi'(x - at) + a^2 \cdot \psi''(x + at).$$

Substituting the second derivatives in equation (7.1) we get the identity. The next problem is to define the unknown functions satisfying the initial conditions (7.4) and (7.5). Let assume $t = 0$. It follows from (7.4)

$$\varphi(x) + \psi(x) = f(x). \qquad (7.21)$$

Taking $t = 0$ in the expression for u'_t we obtain from initial condition (7.5)

$$-a \cdot \varphi'(x) + a \cdot \psi'(x) = F(x). \qquad (7.22)$$

Integrating both sides from 0 to x, we get

$$-\varphi(x) + \psi(x) = \frac{1}{a}\int_0^x F(x)dx + C, \qquad (7.23)$$

C is a constant. It follows from the system of equations (7.21) and (7.23)

$$\varphi(x) = \frac{1}{2} f(x) - \frac{1}{2a} \cdot \int_0^x F(x)dx - \frac{C}{2},$$
$$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2a} \cdot \int_0^x F(x)dx + \frac{C}{2}.$$

Taking into account equality (7.22) and changing argument x on $x - at$ and $x + at$ we find the function $u(x,t)$

$$u(x,t) = \frac{f(x - at) + f(x + at)}{2} + \frac{1}{2 \cdot a} \cdot \int_{x-at}^{x+at} F(x)dx. \qquad (7.24)$$

This formula is called **the D'alembert's solution of the Cauchy's problem for wave equation .**

Example.

Solve the Cauchy's problem for equation (7.1) under the next initial conditions

$$u(x,0) = e^{-x^2},$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0.$$

Taking into account equalities $f(x) = e^{-x^2}$, $F(x) = 0$, we get the answer

$$u(x,t) = \frac{e^{-(x-at)^2} + e^{-(x+at)^2}}{2}.$$

The deflection of the endless string in time according to the answer is shown on the figure 4. It is the sum of two running waves. Both waves are the graphs of the function $f(x) = \frac{1}{2} \cdot e^{-x^2}$. The first wave moves on the left, the second wave moves on the right. The rate of movement is equal to a.

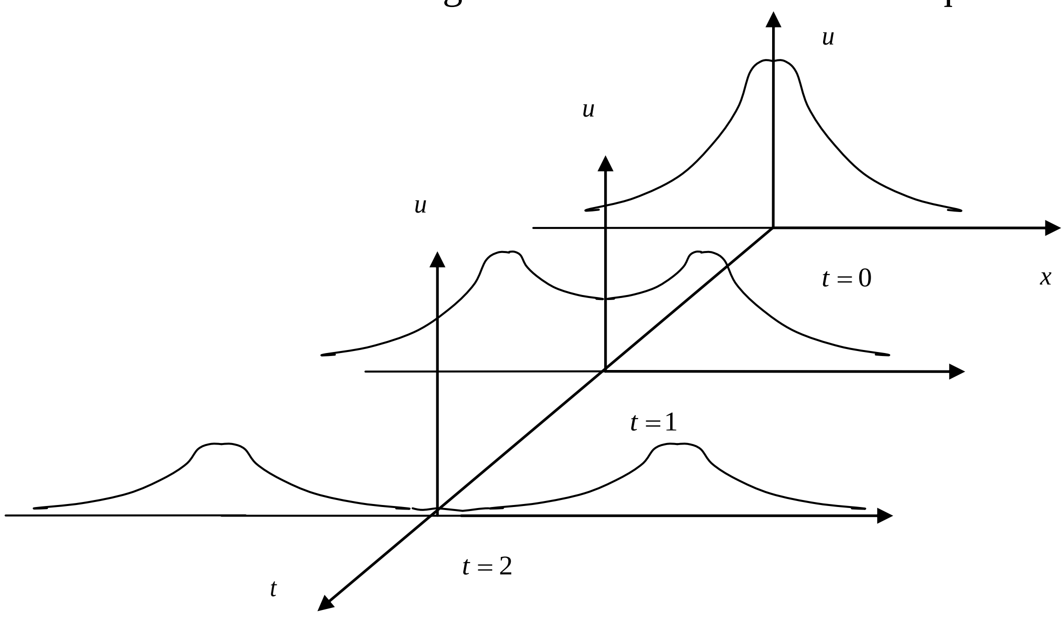


Fig.4. Running waves

7.5. The equation of heat conduction in a rod. Formulation of the boundary-value problem

Let us consider a homogeneous rod of length l. Let us assume that the lateral surface of the rod is impenetrable to heat transfer and the temperature is the same at all points of any cross-Chapteral area of the rod. Let us study the process of propagation of heat in the rod. Let $u(x,t)$ be the temperature in the cross Chapter of the rod with abscissa x at time t. Experiment tells us that the rate of propagation of heat (that is, the quantity of heat passing through a cross Chapter with abscissa x in unit time) is given by the formula

$$q = -k \cdot \frac{\partial u}{\partial x} \cdot S \tag{7.25}$$

where S is the cross-Chapteral area of the rod and k is the coefficient of thermal conductivity $-k\left.\frac{\partial u}{\partial x}\right|_{x=x_2} \cdot S \cdot \Delta t$. The quantity of heat passing through the cross Chapter with abscissa x_1 during time Δt will be equal to

$$\Delta Q_1 = -k\left.\frac{\partial u}{\partial x}\right|_{x=x_1} \cdot S \cdot \Delta t$$

and the same for the cross Chapter with abscissa x_2:

$$\Delta Q_2 = -k\left.\frac{\partial u}{\partial x}\right|_{x=x_2} \cdot S \cdot \Delta t .$$

The influx of heat $\Delta Q_1 - \Delta Q_2$ into the rod element during time Δt will be

$$\Delta Q_1 - \Delta Q_2 = -k\left.\frac{\partial u}{\partial x}\right|_{x=x_1} \cdot S \cdot \Delta t -$$

$$\left(-k\left.\frac{\partial u}{\partial x}\right|_{x=x_2} \cdot S \cdot \Delta t\right) \approx k\left.\frac{\partial^2 u}{\partial x^2}\right|_{x=x_1} \cdot S \cdot \Delta t \cdot \Delta x \tag{7.26}$$

This influx of heat during time Δt was spent in raising the temperature of the rod element by Δu

$$\Delta Q_1 - \Delta Q_2 = c \cdot \rho \cdot \Delta x \cdot S \cdot \Delta u \approx c \cdot \rho \cdot \Delta x \cdot S \cdot \frac{\partial u}{\partial t} \cdot \Delta t \tag{7.27}$$

where c is the heat capacity of the substance of the rod and ρ is the density of the substance. Equating (7.25) and (7.26), we get

$$k \cdot \frac{\partial^2 u}{\partial x^2} \cdot S \cdot \Delta x \cdot \Delta t = c \cdot \rho \cdot \Delta x \cdot S \cdot \frac{\partial u}{\partial t} \cdot \Delta t .$$

Denoting $\frac{k}{c \cdot \rho} = a^2$, we finally get

$$\frac{\partial u}{\partial t} = a^2 \cdot \frac{\partial^2 u}{\partial x^2} . \tag{7.28}$$

This is the equation for the propagation of heat (the equation of heat conduction) in a homogeneous rod.

For the solution of equation (7.28) to be definite, the function $u(x,t)$ must satisfy the boundary-value conditions corresponding to the physical conditions of the problem. For the solution of equation (7.28), the boundary-value conditions may differ. The conditions which correspond to the first boundary-value problem for $0 \le t \le T$ are as follows:

$$u(x,0)=\varphi_1(x) \quad (7.29)$$
$$u(0,t)=\psi_1(t) \quad (7.30)$$
$$u(l,t)=\psi_2(t) \quad (7.31)$$

Condition (7.29) (the initial condition) correspondi to the fact that for $t=0$ the temperature is given in various cross Chapters of the rod and is equal to $\varphi_1(x)$. Conditions (7.30) and (7.31) (the boundary conditions) correspond to the fact that at the ends of the rod, $x=0$ and $x=l$, the temperature is maintained equal to $\psi_1(t)$ and $\psi_2(t)$, respectively.

It is proved that the equation (7.28) has only one solution in the region $0\le x\le l$, $0\le t\le T$, which satisfies the conditions (7.29), (7.30) and (7.31).

7.6. Solving the first boundary-value problem for the heat-conduction equation by the method of finite differences

Let us replace derivatives by appropriate differences

$$\frac{\partial u(x,t)}{\partial x}\approx\frac{u(x+h,t)-u(x,t)}{h}$$

$$\frac{\partial^2 u(x,t)}{\partial x^2}\approx\frac{1}{h}\left(\frac{u(x+h,t)-u(x,t)}{h}-\frac{u(x,t)-u(x-h,t)}{h}\right)$$

or

$$\frac{\partial^2 u(x,t)}{\partial x^2}\approx\frac{u(x+h,t)-2u(x,t)+u(x-h,t)}{h^2} \quad (7.32)$$

similarly,

$$\frac{\partial u(x,t)}{\partial t}\approx\frac{u(x,t+l)-u(x,t)}{l}. \quad (7.33)$$

The first boundary-value problem for the heat conduction equation is stated as follows. It is required to find the solution of the equation (7.28) that satisfies the boundary-value conditions (7.29), (7.30), (7.31), that is, we have to find the solution $u(x,t)$ in a rectangle bounded by the straight lines $t=0,\quad x=0,\quad x=L,\quad t=T$, if the values of the desired function are given on three of its sides: $t=0,\quad x=0,\quad x=L$. We cover region with a grid formed by the straight lines (Fig. 5)

$x=i\cdot h,\quad i=0,1,2,...,$

$t=k\cdot l,\quad k=0,1,2,...$

and approximate the values at the lattice points of the grid, (the points of interChapter of these lines. Introducing the notation $u(ih,kl)=u_{i,k}$. We write a corresponding difference equation for the point (ih,kl). In accord with

(7.32) and (7.33) we get

$$\frac{u_{i,k+1} - u_{i,k}}{l} = a^2 \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{h^2}.$$

We determine

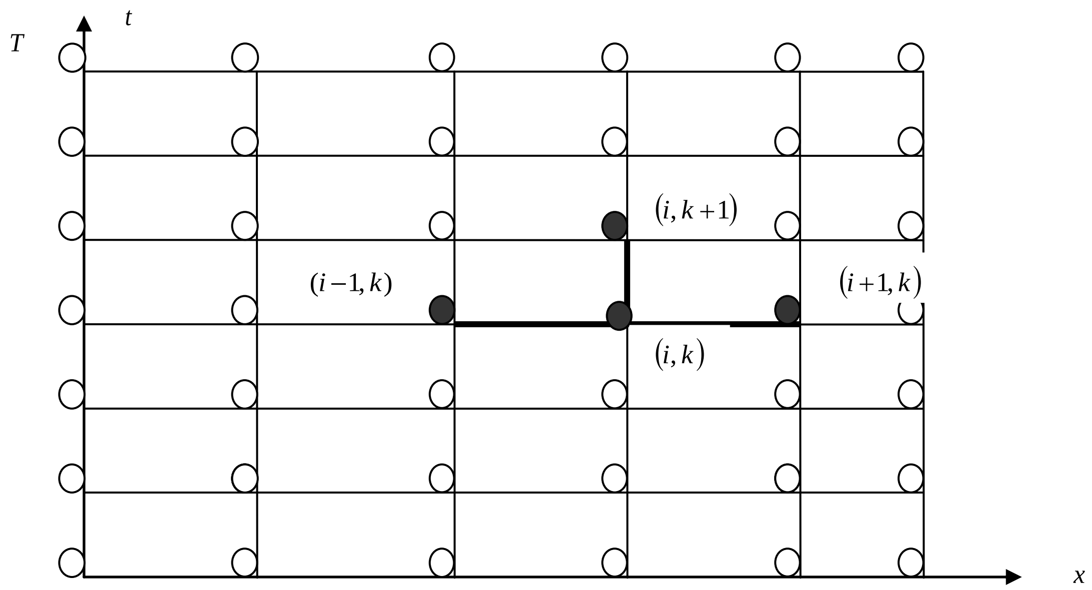


Fig.5. Grid formed by the straight lines

$$u_{i,k+1} = \left(1 - \frac{2a^2 l}{h^2}\right) u_{i,k} + a^2 \frac{l}{h^2}\left(u_{i+1,k} + u_{i-1,k}\right). \tag{7.34}$$

From (7.34) it follows that if we know three values in the row number k, we can determine the value $u_{i,k+1}$ in the $(k+1)$-th row. We know from (7.29) all the values on the straight line $t = 0$. By formula (7.34) determine the values at all the interior points of the segment $t = l$. We know the values at the end points of this segment by virtue of (7.30) and (7.31). In this way, row by row, we determine the values of the desired solution at all lattice points of the grid. It may be proved that from (7.34) we can obtain an approximate value of the solution not for an arbitrary relationship between the steps h and l, but only for $l \le \frac{h^2}{2a^2}$. Formula (7.34) is greatly simplified if the step length l is $l = \frac{h^2}{2a^2}$.

In this case, formula (7.34) takes the form

$$u_{i,k+1} = \frac{1}{2}\left(u_{i+1,k} + u_{i-1,k}\right).$$

Table 33

Basic definitions

English	***Russian***	***Ukrainian***
wave equation	волновое уравнение	хвильове рівняння
transversal vibrations of a string	поперечные колебания струны	поперечні коливання струни
flexible and elastic thread	гибкая упругая нить	гнучка пружна нитка
tension	напряжение	напруження
deflect	отклонение	відхилення
impact	удар	удар
to cancel out	вычеркивать	викреслювати
longitudinal vibrations of rods	продольные колебания стержней	повздовжні коливання стержнів
torsional oscillations of shafts	крутильные колебания валов	крутильні коливання валів
filtration of liquids and gases	фильтрация жидкости и газа	фільтрація рідини та газу
equations of parabolic type	уравнение параболического типа	рівняння параболічного типу
propagation of heat	распространение тепла	поширення тепла
porous medium	пористая среда	пористе середовище
subterranean sandstones	подземні пісчаники	подземні пісчаники
equations of parabolic type	уравнение гиперболического типа	рівняння гіперболічного типу
Laplace equation	уравнение Лапласа	рівняння Лапласа
probability theory	теория вероятностей	теорія ймовірностей
heat-conduction equation	уравнение теплопроводности	рівняння теплопровідності
boundary-value problem	краевая задача	крайова задача
equation of the vibrating string	уравнение колебаний струны	рівняння коливань струни
boundary conditions	предельные условия	граничні умови
initial conditions	начальные условия	початкові умови
boundary-value conditions	краевые условия	граничні умови
method of separation of variables	метод разделения переменных	метод поділу змінних
eigenvalues	собственные числа	власні числа
eigenfunctions	собственные функции	власні функції
double term-by-term differentiation with respect to x and to t	двойное почленное дифференцирование по перменным x и t	подвійне почленне диференціювання за змінними x та t

Chapter 8. ELEMENTS OF THE THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS

It is not sufficient merely to indicate the fact of randomness in order to make use of a particular phenomenon of nature or to control a technological process. We have to learn to evaluate random events numerically and predict the course they will take. Such, at the present time, are the insistent demands of theoretical and practical problems. Two divisions of mathematics are engaged in the solution of such problems and in constructing the requisite general mathematical theory: they are the theory of probability and mathematical statistics.

8.1. Random event. Relative frequency of a random event. The probability of an event. The subject of probability theory

The basic concept of probability theory is that of a ***random (chance) event***. A random event is an event which may occur or fail to occur under the realization of a certain set of conditions.

Example. In coin tossing, the occurrence of heads is a random event.

Example. In firing at a target from a particular gun, hitting the target or a given area on it is a random event.

Definition. ***The relative frequency p**** of a random event A is the ratio of the number m^* of occurrences of the given event to the total number n^* of identical trials, in each of which the given event could occur or fail to occur. We will write

$$P^*(A) = p^* = \frac{m^*}{n^*}.$$

Example. Suppose, under identical conditions, we fire 6 sequences of shots at a given target;

In the first sequence there were 5 shots and 2 hits,

In the second sequence there were 10 shots and 6 hits,

12 shots and 2 hits

50 shots and 27 hits

100 shots and 49 hits

200 shots and 102 hits

Event A is a hit. The relative frequency of hit in the sequences will be 0.40, 0.60, 0.58, 0.54, 0.49, 0.51.

From observations of a variety of phenomena, we can conclude that if the number of trials in each sequence is small, then the relative frequencies of the occurrence of event A in the different sequences can differ substantially from one another. However, if the number of trials in the sequences is great, then, as a rule, the relative frequencies of the occurrence of event *A* in different sequences will differ but slightly, and the difference is the smaller, the greater the number of trials in the sequences. We say that the relative frequency in a large number of trials ceases more and more to be accidental (of a random nature). Experiments show that in most cases there is a constant *p* such that the relative frequencies of occurrence of an event *A*, given a large number of trials, differ but slightly from *p,* except in rare cases.

This experimental fact is symbolized as follows:

$$\frac{m^{\bullet}}{n^{\bullet}} \xrightarrow[n^{\bullet}]{} p$$

The number *p* is called the *probability* of occurrence of a random event *A*. This statement is symbolized as

$$P(A) = p$$

The probability *p* is an objective characteristic of the possibility of occurrence of event *A* under given trials. It is determined by the nature of *A*.

Given a large number of trials, the relative frequency differs very slightly from the probability, except in rare cases, which may be ignored.

Since probability is an objective characteristic of the possibility of occurrence of a certain event, to predict the course of numerous processes that one has to consider in military affairs, in the organization of production, in economic situations, etc., it is necessary to be able to determine the probability of occurrence of certain compound events. Determining the probability of occurrence of an event on the basis of the probabilities of the elementary events governing the given compound event, and the study of probabilistic regularities of various random events constitute the subject of the *theory of probability*.

8.2. The classical definition of probability and the calculation of probabilities

In many cases it is possible to calculate the probability of a random event by proceeding from an analysis of the trial.

Example. A homogeneous cube with faces labeled 1 to 6 is called a die. We will consider the random event of the occurrence of a number *l* $(1 \le l \le 6)$ on the upper face for each throw of the die. By virtue of the symmetry of the die, the events (the appearance of any number from 1 to 6) are equally probable. Hence they are called ***equally probable events***. Given a large number of throws, *n* it can be expected that the number *l* (and any other number from 1 to 6) will turn up in roughly $n/6$ cases. Experiments corroborate this fact.

The relative frequency will be close to the number $p^{\bullet} = n/6$. It is therefore considered that the probability of the number l $(1 \le l \le 6)$ turning up is equal to 1/6.

Definition. Random events in a given trial are called ***disjoint (mutually exclusive)*** if no two can occur at the same time.

Definition. We will say that random events form a ***complete group*** if in each trial any one of them can occur but no disjoint event can occur.

We consider **a complete group of equally probable disjoint random events.** We give the name *cases* to such events. An event (case) of such a group is termed *favorable* to the occurrence of event *A* if the occurrence of the case implies the occurrence of *A*.

Example. We have 8 balls in an urn. Each ball is numbered from 1 to 8. Balls labeled 1, 2, 3 are red, the others are black. The occurrence of a ball labeled 1 (or 2 or 3) is an event favorable to the occurrence of a red ball.

For this case, we can give a definition of probability that differs from that given above.

Definition. The ***probability p*** of event *A* is the ratio of the number *m* of favorable cases to the number *n* of all possible cases forming a complete group of equally probable disjoint events, or, symbolically,

$$P(A) = p = \frac{m}{n}$$

Definition. If relative to some event there are *n* favorable cases forming a complete group of equally probable disjoint events, then such an event is called a ***certain event***. A certain event has probability $p=1$.

If not a single one of *n* cases forming a complete group of equally probable disjoint events is favorable to an event, then it is termed an ***impossible event*** and has probability $p = 0$. From the definition of probability it follows that the relation

$$0 \le p \le 1$$

holds true.

Example. Ten items out of a set of 100 are defective. What is the probability that 3 out of any 4 chosen items will not be defective?

Solution. Four items out of 100 can be chosen in the following number of ways: $n = C_{100}^{4}$. The number of cases where 3 out of 4 items are nondefective is equal to $m = C_{90}^{3} \cdot C_{10}^{1}$. The desired probability is

$$p = \frac{m}{n} = \frac{C_{90}^{3} \cdot C_{10}^{1}}{C_{100}^{4}} \approx 0.3 .$$

8.3. The addition of probabilities. Complementary random events

Definition. *The logical sum (union)* of two events A_1 and A_2 is an event *C* consisting in the occurrence of at least one of the events.

Let us consider the probability of the union of two disjoint events A_1 and A_2. The union of these events is denoted by

$A_1 + A_2$

The following theorem, which is called the *theorem on the addition of probabilities*, holds true.

Theorem1. ***Suppose, in a given trial (phenomenon, experiment), a random event A_1 can occur with probability $P(A_1)$ and an event A_2 with probability $P(A_2)$. The events A_1 and A_2 are exclusive. Then the probability of the union of the events, that is, the probability that either event A_1 or event A_2 will take place, is computed from the formula***

$$P(A_1 or A_2) = P(A_1) + P(A_2) \qquad (8.1)$$

The proof of this theorem is the same for any number of terms:

$$P(A_1 or A_2 or ... or A_n) = P(A_1) + P(A_2) + ... + P(A_n)$$

Definition. Two events are called ***complementary events*** if they are exclusive and form a complete group.

If one event is denoted by A, the complement (complementary event) is denoted by $\overline{A}$. Let the probability of the occurrence of event *A* be *p*, the probability of the nonoccurrence of event *A*, that is, the probability of the occurrence of event $\overline{A}$, be $P(\overline{A}) = q$. On trial, either A or $\overline{A}$ will occur, therefore Theorem 1 gives

$$P(A) + P(\overline{A}) = 1. \qquad (8.2)$$

That is***, the union of the probabilities of complementary events is equal to unity****:*

$p + q = 1$

Corollary. ***If random events $A_1, A_2, ..., A_n$ form a complete group of exclusive events, then the following equation holds true****:*

$$P(A_1) + P(A_2) + ... + P(A_n) = 1 \qquad (8.3)$$

Definition. Random events A and *B* are called ***compatible*** if in a given trial both events can occur, which is to say we have a ***logical product (interChapter)*** of events *A* and *B*.

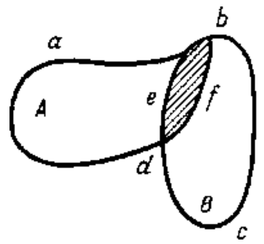
The event which consists in the interChapter of *A* and *B* is denoted by (*A* and *B)* or *(AB).* The probability of the interChapter of events *A* and *B* will be denoted by P(*A* and *B)*.

Theorem. ***The probability of the union of compatible events is computed from the formula***

$$P(A \quad or \quad B) = P(A) + P(B) - P(A \quad and \quad B) \qquad (8.4)$$

The truth of formula (1.4) can be illustrated geometrically. We first give the definition.

Definition. Given a certain domain *D* with area *S*. Consider a subdomain *d* of *D*. Let S_1 be the area of *d*. Then the probability of a point falling in *d* (the falling of a point in *D* is taken to be a certain event) is, by definition, S_1/S, or $p=S_1/S$. This is called ***geometric probability***.



$P(A \quad or \quad B) = area \ \ abcda$

$P(A) = area \quad abfda$

$P(B) = area \quad bcdeb$

$P(A \quad and \quad B) = area \ debfd$

8.4. Multiplication of probabilities of independent events

Definition. An event *A* is said to be *independent* of *B* if the probability of occurrence of *A* does not depend on whether event *B* took place or not.

Theorem. *If random events A and B are independent, then the probability of the interChapter of events A and B is equal to the product of the probabilities of occurrence of A and B:*

$$P(A \quad and \quad B) = P(A) \cdot P(B) \qquad (8.5)$$

8.5. Dependent events. Conditional probability. Total probability

Definition. Event *A* is said to be *dependent* on event *B* if the probability of occurrence of *A* depends on whether *B* took place or not.

The probability that event *A* occurred, provided that B took place, will be denoted by $P_B(A)$ and will be called the conditional probability of event A provided that B has occurred.

Theorem. ***The probability of the interChapter of two events is equal to the product (logical interChapter) of the probability of one by the conditional probability of the other computed on the condition that the first event has taken place****, that is*

$$P(A \quad and \quad B) = P(B) \cdot P_B(A) \qquad (8.6)$$

Total probability

Theorem. ***If event A can be realized only when one of the events*** B_1***,*** B_2***, ..., B_n, which form a complete group of exclusive events, occurs, the probability of event A is computed from the formula***

$$P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + ... + P(B_n)\cdot P_{B_n}(A) \quad (8.7)$$

Formula (8.7) is called the **formula of total probability**.

Proof. Event A can occur if one of the compatible events (B_1 and A), (B_2 and A), ..., (B_n and A)

is realized. Consequently, by the theorem of addition of probabilities, we get $P(A)=P(B_l \text{ and } A)+P(B_2 \text{ and } A)+\ldots+P(B_n \text{ and } A)$

Replacing the terms of the right side in accordance with formula (8.1), we get equation (8.7).

8.6. Probability of causes. Bayes's formula

Statement of the problem. We will consider a complete group of exclusive events $B_1, B_2, ..., B_n$, the probabilities of occurrence of which are $P(B_1), P(B_2), ..., P(B_n)$. Event *A* can occur only together with some one of the events $B_1, B_2, ..., B_n$, which we will call ***causes.***

The probability of the occurrence of event *A* is, in accord with formula (8.8)

$$P(A) = P(B_1)\cdot P_{B_1}(A) + P(B_2)\cdot P_{B_2}(A) + ... + P(B_n)\cdot P_{B_n}(A). \quad (8.8)$$

Suppose that event *A* has taken place. This fact will alter the probability of the causes, $P(B_1), P(B_2), ..., P(B_n)$. It is required to determine the conditional probabilities of the realization of these causes on the assumption that event *A* has occurred, that is, to determine $P_A(B_1), P_A(B_2), ..., P_A(B_n)$.

(alter $[o:lte]$ - изменять)

Solution of the problem. We will find the probability *P (A* and *B₁):*

$$P(A \quad and \quad B_1) = P(B_1)\cdot P_{B_1}(A) = P(A)\cdot P_A(B_1)$$

hence

$$P_A(B_1) = \frac{P(B_1)\cdot P_{B_1}(A)}{P(A)}.$$

Substituting for P *(A)* its expression (8.8), we get

$$P_A(B_1) = \frac{P(B_1)\cdot P_{B_1}(A)}{\sum_{i=1}^{n} P(B_i)\cdot P_{B_i}(A)}. \quad (8.9)$$

The probabilities $P_A(B_2), P_A(B_3), ..., P_A(B_n)$ are determined in similar fashion:

$$P_A(B_k) = \frac{P(B_k)\cdot P_{B_k}(A)}{\sum_{i=1}^{n} P(B_i)\cdot P_{B_i}(A)}.$$

$P_A(B_k)$ - the probability of the realization of cause B_k provided that event *A* has occurred.

Formula (8.9) is called ***Bayes's formula*** or the ***theorem of causes***. ***(Bayes' s rule for the probability of causes.)***

8.7. The Bernoulli's scheme of the repeated trials

Suppose we have a sequence of n trials, in each of which event A can occur with probability p.

Theorem. ***The probability*** $P_n(m)$ ***that in n trials event A will occur m times and the event*** $\overline{A}$ ***(nonoccurrence of A) will occur n-m times is equal to*** $C_n^m \cdot p^m q^{n-m}$***, where*** C_n^m ***is the number of combinations of n elements taken m at a time; p is the probability of the occurrence of event A,* p=P(*A*);** ***q is the probability of the nonoccurrence of event A, that is*** $q = 1 - p = P(\overline{A})$**.**

$$P_n(m) = C_n^m \cdot p^m q^{n-m}$$

8.8. A discrete random variable. The distribution law of a discrete random variable

Definition. A variable quantity X which, in a trial, assumes one value out of a finite or infinite sequence $x_1, x_2, \ldots, x_k, \ldots$ is called a **discrete random quantity** (or variable), if to each value x_k there corresponds a definite probability p_k that the variable x will assume the value x_k.

It follows from the definition that to every value x_k there corresponds a probability p_k.

The functional relationship between p_k and x_k is called the **distribution law of probabilities of a discrete random variable** x *

Possible values of the random variable X	x_1	x_2	…	x_i	…	x_n
Probabilities of these values p	p_1	p_2	…	p_i	…	p_n

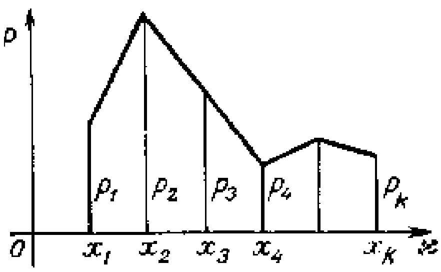


Fig. 160

Fig. 6.

The distribution law can also be represented graphically in the form of a **polygon of probability distribution** (also called a **frequency polygon**): in a rectangular coordinate system, points are constructed with coordinates (x_k, p_k) and are joined by a polygonal line.

8.9. Relative frequency and the probability of relative frequency in repeated trials

Let X be a random variable denoting the relative frequency of occurrence of event *A* in athe sequence consisting of *n* trials. The probability $P\left(x=\frac{m}{n}\right)$ that the random variable X will assume the value $\frac{m}{n}$, that is, that in *n* trials event *A* will occur *m* times and the event $\overline{A}$ (nonoccurrence of *A*) will occur *n-m* times is equal to $C_n^m p^m q^{n-m}$, where C_n^m is the number of combinations of n elements taken *m* at a time; *p* is the probability of the occurrence of event *A*, $p = P(A)$; *q* is the probability of the nonoccurrence of event *A*, that is, $q=1—p = P(\overline{A})$. Let's make the distribution law. This distribution law is known as the **binomial distribution** because the probabilities p_i are equal to the corresponding terms in the binomial expansion of the expression $(q — p)^n$.

8.10. The mathematical expectation of a discrete random variable

Definition The **mathematical expectation** (or, simply, expectation) of a random variable X (we symbolize expectation by $M(X)$ is the sum of the products of all possible values of the random variable by the probabilities of its values.

$$M(X)=\sum_{k=1}^{n} x_k p_k .$$

In a large number of trials, the arithmetic mean of the observed values is close to the expectation; or the arithmetic mean of the observed values of a random variable tends to the expectation when the number of trials increases without bound.

Variance. Root-mean-square (standard) deviation

In addition to the expectation of a random variable X, which defines the position of the centre of a probability distribution, a distribution is further characterized quantitatively by the **variance** of the random variable. The variance is denoted by $D(X)$.

The word *variance* means dispersion. Variance is a numerical characteristic of the dispersion, or spread of values, of a random variable about its mathematical expectation.

Definition. The **variance** of a random variable X is the expectation of the square of the difference between X and its expectation (that is, the expectation of the square of the appropriate centred random variable.

$$D(X) = M\left((x - m_x)^2\right) \quad \text{or} \quad D(X) = \sum_{k=1}^{n} (x_k - m_k)^2 p_k . \qquad (8.10)$$

Variance has the dimensions of the square of the random variable. It is sometimes more convenient in describing dispersion to make use of a quantity whose dimensions coincide with those of the random variable. This quantity is termed the **root-mean-square deviation (standard deviation)**

Definition. The **root-mean-square deviation (standard deviation)** is the square root from the variance.

$\sigma(X) = \sqrt{D(X)}$.

Note. In computing variance, it is often convenient to transform formula (8.10) as follows

$D(X) = M(X^2) - m_x^2$.

Properties of the mathematical expectation and the variance of a discrete random variable

1. $M(c) = c$, $(c - const)$
2. $M(c \cdot X) = c \cdot M(X)$,
3. $M(X + Y) = M(X) + M(Y)$,
4. $M(X \cdot Y) = M(X) \cdot M(Y)$,
5. $D(c) = 0$,
6. $D(c \cdot X) = c^2 \cdot D(X)$,
7. $D(X + Y) = D(X) + D(Y)$,
8. $D(X - Y) = D(X) + D(Y)$.

8.11. Continuous random variable. Probability density function of a continuous random. Variable. The probability of the random variable. Falling in a specified interval

Example. The amount of wear of a cylinder is measured after a certain period of operation. This quantity is determined by the value of the increase in diameter of the cylinder. We denote it by x. From the essence of the problem, it follows that x can assume any value in a certain interval *(a, b)* of possible values. This quantity is termed a **continuous random variable**.

We consider the continuous random variable x specified on a certain interval (a,b) which can also be an infinite interval, $(-\infty,+\infty)$. We divide this interval into subintervals of length $\Delta x_{i-1} = x_i - x_{i-1}$ by the arbitrary points $x_0, x_1, x_2, ..., x_n$.

Suppose we know the probability that the random variable x will fall in the interval $(x_i - x_{i-1})$. We denote this probability by $P(x_{i-1} < x < x_i)$ and represent it as the area of a rectangle with base Δx_i AAV (Fig. 6).

Definition. If there exists a function $y = f(x)$ such that

$$\lim_{\Delta x \to 0} \frac{P(x < \varepsilon < x + \Delta x)}{\Delta x} = f(x) \quad (8.11)$$

then this function $f(x)$ is termed the ***probability density_function of the random variable x*** (or, simply, ***density function),*** or the distribution.

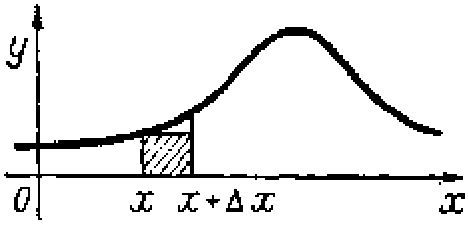


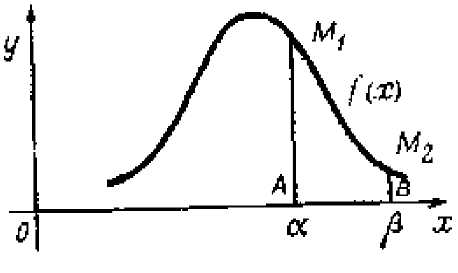
Fig. 7.

(It is also called the ***frequency function, distribution density***, or the ***probability density**.)* We will use X to denote the continuous random variable, *x* or x_k to denote the values of this random variable. The curve *y = f(x)* is called the ***probability curve*** or the ***distribution curve*** (Fig. 7). Using the definition of limit, from equation (8.12) follows, to within infinitesimals of higher order than Δx, the approximate equation

$$P(x < X < x + \Delta x) \approx f(x) \cdot \Delta x \quad (8.12)$$

The following theorem holds true.

Theorem. Let $f(x)$ be the density function of the random variable x. Then the probability that a value of the random variable x will fall in some interval (α, β) is equal to the definite integral of the function $f(x)$ from α to β that is, we have the following equation:



$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f(x)dx \quad (8.13)$$

Fig. 8.

Knowing the probability density function of a random variable, we can determine the probability that a value of the random variable will lie in a given interval. Geometrically, this probability is equal to the area of the resulting curvilinear trapezoid (Fig. 8).

It is possible to verify that $\int_{-\infty}^{+\infty} f(x)dx = 1$.

8.12. The distribution function

Definition. Let $f(x)$ be the density function of some random variable x $(-\infty < x < +\infty)$; then the function

$$F(x) = \int_{-\infty}^{x} f(x)dx \tag{8.14}$$

is called the **distribution function.**

From equation (8.13), it follows that the distribution function $F(x)$ is the probability that the random variable x will assume a value less than x.

The value of the distribution function for a given value of x is numerically equal to the area bounded by the distribution curve lying to the left of the ordinate drawn through the point x. The graph of the function $F(x)$ is termed the probability distribution curve.

Theorem. The probability of a random variable x lying in a given interval (α, β) is equal to the increment in the distribution function over that interval:

$$P(\alpha < X < \beta) = F(\beta) - F(\alpha).$$

Note that the density function $f(x)$ and the corresponding distribution function $F(x)$ are connected by the relation $F'(x) = f(x)$.

This follows from (8.4) and the theorem on differentiating a definite integral with respect to the upper limit. It can be shown $F(x)$ increases when x increases and $0 < F(x) < 1$.

8.13. Numerical characteristics of a continuous random variable

Let us examine the numerical characteristics of a continuous random variable x with density function $f(x)$.

Definition The **mathematical expectation** of a continuous random variable x with density function $f(x)$ is the expression

$$M(x) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx.$$

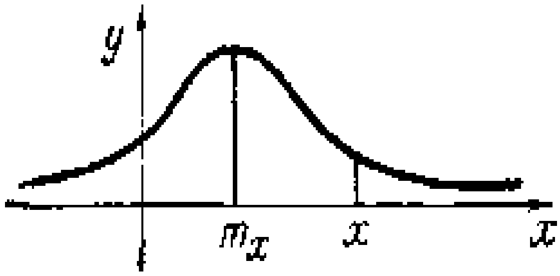


Fig. 9.

It is possible to use the symbol m_x for the expectation. The expectation is called the **centre of probability distribution** of the random variable. (Fig. 9). If the distribution curve is symmetric about the x-axis, that is, if $f(x)$ is an even function, then clearly

$$M(x)=\int_{-\infty}^{\infty} x\cdot f(x)\cdot dx=0$$

Let us consider a centered random variable $x-m_x$. We will find its expectation

$$M(x-m_x)=\int_{-\infty}^{\infty}(x-m_x)\cdot f(x)\cdot dx=0$$

The expectation of a centered random variable is zero.

Definition. The ***variance*** of a random variable x is the expectation of the square of the corresponding centred random variable

$$D(x)=\int_{-\infty}^{\infty}(x-m_x)^2\cdot f(x)\cdot dx.$$

Definition. The **standard deviation** of a random variable x is the square root of the variance:

$$\sigma(x)=\sqrt{D(x)}.$$

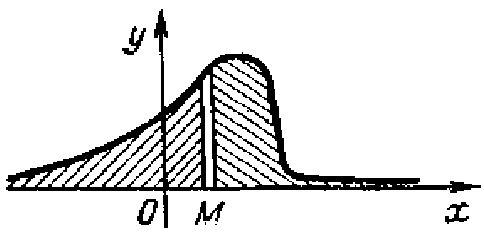


Fig. 10.

Definition. The value of the random variable x for which the density function is a maximum is termed the **mode** *(*symbolized by M). For the centered random variable x the mode coincides with the expectation.

Definition. A number (symbolized by M_e*)* is called the **median** (Fig.10), if it satisfies the equation

$$\int_{-\infty}^{M_e} f(x)\cdot dx = \int_{M_e}^{+\infty} f(x)\cdot dx = \frac{1}{2}.$$

8.14. Normal distribution the expectation of a normal distribution

Studies of various phenomena show that many random variables, such, for example, as measurement errors, the lateral deviation and range deviation of the point of impact from a certain centre in gunfire, and the amount of wear in machine parts, have a density function given by the formula

$$f(x) = \frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right) \qquad (8.15)$$

We say the random variable has **normal distribution** or is **normally distributed** (the term **Gaussian distribution** is also used). The so-called normal curve (normal distribution curve) in shown at the fig. 11.

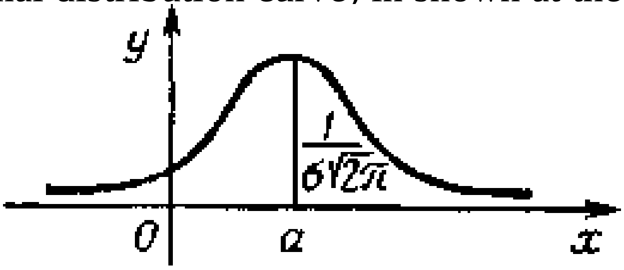


Fig. 11

It can be shown that the density function (8.15) satisfies the basic relation $\int_{-\infty}^{+\infty} f(x)dx = 1$.

The expectation of a random variable with normal distribution is

$$m_x = \int_{-\infty}^{+\infty} x\cdot\frac{1}{\sigma\cdot\sqrt{2\pi}}\cdot\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right)dx = a.$$

The value of the parameter a in formula (8.15) is equal to the expectation of the random variable under consideration. The point $x = a$ is the centre of the distribution or the centre of dispersion. When $x = a$ the function $f(x)$ has a maximum and so the value $x = a$ is the *mode* of the random variable. It may be shown that the median of the normal distribution is equal to a.

8.15. Variance and standard deviation of a normally distributed random variable

The variance of a continuous random variable is found by formula

$$D(x) = \int_{-\infty}^{+\infty} x^2 \cdot \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{(x-a)^2}{2\sigma^2} \right) dx .$$

Calculation gives the result

$$D(x) = \sigma^2 .$$

The standard deviation, in accordance with formula

$\sigma(x) = \sqrt{D(x)}$ is $\sigma(x) = \sigma$.

Thus, the variance is equal to the parameter σ^2 in the density function formula (8.15).

8.16. The probability of a value of the random variable falling in a given interval. The Laplace function. Normal distribution function

Let us determine the probability that a value of the random variable *x* having the density function

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{(x-a)^2}{2\sigma^2} \right)$$

fall in the interval (α, β)

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} f(x) dx$$ or

$$P(\alpha < X < \beta) = \int_{\alpha}^{\beta} \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \exp\left(-\frac{(x-a)^2}{2\sigma^2} \right) dx .$$

Making the change of variable

$$\frac{x-a}{\sigma\sqrt{2}} = t$$

we get

$$P(\alpha < X < \beta) = \frac{1}{\sqrt{\pi}} \int_{\frac{\alpha-a}{\sigma\sqrt{2}}}^{\frac{\beta-a}{\sigma\sqrt{2}}} e^{-t^2} dt . \qquad (8.16)$$

The integral on the right is not expressible in terms of elementary functions. The values of this integral can be expressed in terms of the values of the **probability integral** $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

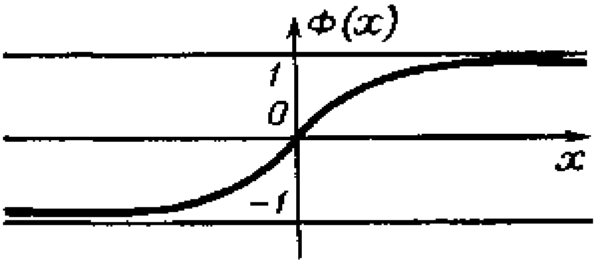


Fig.12.

Here are some of the properties of the function.

1. $\Phi(x)$ is defined for all values of x.
2. $\Phi(0)=0$.
3. $\Phi(+\infty)=1$.
4. $\Phi(x)$ is monotonic increasing on the interval $(0,+\infty)$.
5. $\Phi(x)$ is an odd function since

$\Phi(-x)=-\Phi(x)$.

The graph of the function $\Phi(x)$ is shown in Fig. 12.

Rewrite equation (8.16) using the theorem on the partition of the interval of integration

$$P(\alpha < X < \beta)=\frac{1}{\sqrt{\pi}}\int_{\frac{\alpha-a}{\sigma\sqrt{2}}}^{0} e^{-t^2}dt+\frac{1}{\sqrt{\pi}}\int_{0}^{\frac{\beta-a}{\sigma\sqrt{2}}} e^{-t^2}dt=\frac{1}{2}\cdot\left[\Phi\left(\frac{\beta-a}{\sigma\sqrt{2}}\right)-\Phi\left(\frac{\alpha-a}{\sigma\sqrt{2}}\right)\right] \qquad (8.17)$$

Let us compute the probability that a value of the random variable will fall in the interval $(a-l, a+l)$ symmetric about the point $x=a$.

Formula (8.17) then takes the form

$$P(a-l < X < a+l)=\frac{1}{2}\cdot\left[\Phi\left(\frac{l}{\sigma\sqrt{2}}\right)-\Phi\left(\frac{-l}{\sigma\sqrt{2}}\right)\right]$$

or

$$P(a-l < X < a+l)=\Phi\left(\frac{l}{\sigma\sqrt{2}}\right).$$

The right side does not depend on the position of the centre of dispersion, and so for $a=0$ as well we get

$$P(-l < X < +l)=\Phi\left(\frac{l}{\sigma\sqrt{2}}\right). \qquad (8.18)$$

8.17. The three-sigma rule. Error distribution

In practical computations, the unit of measurement of the deviation of a normally distributed random variable from the centre of dispersion (the mathematical expectation) is taken to be the root-mean-square (standard) deviation σ. Then, by formula (5.18), we get a useful equation:

$$P(-3\sigma < X < +3\sigma) = \Phi\left(\frac{3}{\sqrt{2}}\right) = 0.997.$$

We can be almost certain that the random variable (error) will not depart from the absolute value of the expectation by more than 3σ. This proposition is called the **three-sigma rule.**

Note In place of the function, $\Phi(x)$ frequent use is made of the Laplace function

$$\overline{\Phi}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-\frac{t^2}{2}} dt.$$

The Laplace function is connected with the function $\Phi(x)$ by a simple relation:

$$\overline{\Phi}(x) = \frac{1}{2} \cdot \Phi\left(\frac{x}{\sqrt{2}}\right).$$

PROBLEMS

1) The classical definition of probability

1. One card is drawn from a deck of 36 cards. What is the probability of drawing a spade?

2. Two coins are tossed at the same time. What is the probability of getting 2 heads.

3. Two dice are thrown at one time. Find the probability that a sum of 5 will turn up.

4. One hundred cards are numbered from 1 to 100. Find the probability that randomly chosen card has the digit 5.

5. There are 10 tickets in a lottery: 5 wins and 5 looses. Take two tickets. What is the probability of a win?

6.* A die is thrown 5 times. What is the probability that a six will turn up twice and non-six three times?

7. Ten times out of a set of 100 are defective. What is the probability that 3 out of any 4 chosen items will not be defective?

2) The addition of probabilities. Multiplication of probabilities

8. The probability of hitting a target from one gun is 0.8, from another gun, 0.7. Find the probability of destroying the target in a simultaneous firing from both guns. The target will be destroyed if at least one of guns makes a hit.

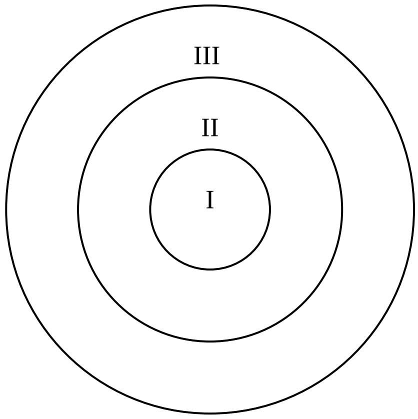
9. Shots are fired at a certain domain D consisting of three non-overlapping zones. The probability of hitting of:

Zone I: $P(A_1) = 0.05$

Zone II: $P(A_2) = 0.1$

Zone III: $P(A_3) = 0.17$

What is the probability of hitting D?



10. Non-failure operation of a device is determined by trouble-free operation of each of three component units. The probabilities of no-failure operation of the units during a certain cycle are $p_1 = 0.6$, $p_2 = 0.7$, $p_3 = 0.9$. Find the probability that the device will not break down the indicated operation cycle.

11. Two tanks are firing at one and the same target. Tank one has a probability of 9/10 of hitting the target. Tank two - a probability of 5/6. One shot is fired from each tank at the same time. Determine the probability that two hits will be scored.

12.* The probability of destroying a target in one shot is equal to *p*. Determine the number *n* of shots needed to destroy the target with probability greater than or equal to *a*?

13. There are 4 machines. The probability that a machine is in operation at an arbitrary time *t* is equal to 0.9. Find the probability that at time t at least one machine is working.

14. The probability of hitting a target is *p*=0.9. Find the probability that in three shots there will be three hits.

15. Box one contains 30% first-grade articles. One article is drawn from each box. Find the probability that both drawn articles are first-grade.

16. The probability of a hit in a single shot is *p*=0.6. Determine the probability that three shots will yield at least one hit.

3) Dependent event. Conditional probability. Total probability. Bayes's formula.

17. The probability of manufacturing a non-defective (acceptable) item by a given machine is equal to 0.9. The probability of the occurrence of quality articles of grade one among the non-defective items is 0.8. Determine the probability of turning out grade-one articles by this machine.

18. Three shots are fired at a target in succession. The probability of a hit in the first shot is $p_1 = 0.3$, in the second, $p_2 = 0.6$, in the third, $p_3 = 0.8$. In the case of one hit, the probability of destroying the target is $\lambda_1 = 0.4$, in the case of two hits, $\lambda_2 = 0.7$, in the case of three hits $\lambda_3 = 1.0$. Determine the probability of destroying the target in three shots.

19. Out of a total of 350 machines, there are 160 of grade one, 110 of grade two, and 80 of grade three. The probability of defectives in the grade-one category is 0.01, in the grade-two category, 0.02, in the grade-three category, 0.04. Take one machine. Determine the probability that it is acceptable.

20. At a factory, 30% of the instruments are assembled by specialists of high qualification, 70% by those of medium qualification. The reliability of an instrument assembled by the former is 0.9, that assembled by the latter, 0.8. An instrument picked off the shelf turns out to be reliable. Determine the probability that it was assembled by the specialists of higher qualification.

21. Stack of two tanks fired independently at a target. The probability of the first tank destroying the target is $p_1 = 0.8$, that of the second, $p_2 = 0.4$. The target is destroyed by a single hit. Determine the probability it was destroyed by the first tank.

4) Repeated trials

22. What is the probability that event A will occur twice (a) in two trials, (b) in three trials, (c) in 10 trials, if the probability of the occurrence of the event in each trial is equal to 0.4?

23. Five independent shots are fired at a target. The probability of a hit is each shot in 0.2. Three hits suffice to destroy the target. Determine the probability of target destruction.

24. Four independent trials are carried out. The probability of the occurrence of event A in each trial is 0.5. Determine the probability that A will occur at least twice.

25. The probability of defective items in a given batch is p=0.1. What is the probability that in a batch of three items there will be 2 defective items?

26. Find the probability of obtaining at least one hit in the case of 10 shots if the probability of hitting the target in a single shot is p=0.15.

Table 34

Basic definitions

English	***Russian***	***Ukrainian***
Theory of Probability	Теория вероятностей	Теорія ймовірностей
random	случайный	випадковий
event	событие	
trial	испытание	випробування
occur	происходить	відбуватися
occurrence	наступление	настання
toss	подбрасывать	підкидати
head	герб	герб
relative frequency	относительная частота	відносна частота
cease	прекращать	припиняти
accidental	случайный	випадковий
compound	составной	складовій
corroborate	подтверждать	підтверджувати
die	игральная кость	гральна кістка
equally probable events	равновозможные события	рівноможліві події
disjoint (mutually exclusive)	несовместный	неспільний
complete group	полная группа	повна група
favorable	благоприятный	спріятливий
certain event	достоверное событие	достовірна подія
impossible event	невозможное событие	неможлива подія
complementary events	противоположные события	протилежні події
compatible events	совместные события	спільні події
urn model	схема урн	схема урн
the logical sum (union) of events	сумма (объединение) событий	сума (об'єднання) подій
the logical product (interChapter) of events	умножение (пересечение) событий	множення (перетинання) подій
conditional probability	условная вероятность	умовна ймовірність
cause	полная вероятность	повна ймовірність
discrete	дискретный	дискретний
distribution law	закон распределения	закон розподілу
frequency polygon	полигон частот	полігон частот
repeated trials	повторные испытания	повторні випробування
mathematical expectation	математическое ожидание	математичне очікування
variance	дисперсия	дисперсія
root-mean-square	среднеквадратическое отклонение	середнеквадратічне відхилення
probability density function	функция плотности вероятностей	функція щільності ймовірностей

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