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THEORY OF PROBABILITY AND MATHEMATICAL STATISTICS: EXAMPLES AND PROBLEMS

TUTORIAL FOR STUDENTS OF ECONOMIC SPECIALTIES

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У посібнику розглянуто приклади розв'язання практичних завдань з теорії ймовірностей та математичної статистики, наведені завдання для самостійного виконання та питання для перевірки власних знань. Наприкінці кожного розділу є перелік рекомендованої літератури. Призначається для англомовних студентів економічних спеціальностей.

Thee tutorial includes examples of solving practical problems in the theory of probability and mathematical statistics, tasks for independent performance and questions for testing one's own knowledge. At the end of each chapter there is a list of recommended literature. It is intended for English-speaking students of economic specialties.

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CONTENTS

INTRODUCTION

This tutorial is intended for second-year students of economic specialties. The material of the tutorial has a clear logical structure. It consists of eight sections, each of which includes examples of solving practical problems, tasks for self-solving, test questions for self-test and recommended literature. In the first section problems of combinatorics are considered, in the second section problems of the classical definition of probability are analyzed, the third section is devoted to problems of the total probability and Bayes' formulas, in the fourth section problems related to repeated trials are considered. The fifth, sixth and seventh sections present problems related to the calculation of the numerical characteristics of random variables, finding the distribution function, distribution density, plotting their graphs. In the eighth (final section) the simplest problems of mathematical statistics are considered.

It should be noted that all components of the tutorial harmoniously complement each other, create the most favorable conditions for the best assimilation of the material. Working with the tutorial, the student learns to solve practical problems, checks his knowledge using tasks for self-test and finally consolidates the acquired knowledge and skills by solving problems on his own. The recommended literature at the end of each section includes both printed and electronic publications. Working with literature teaches students to think independently, analyze the information, draw conclusions, develops the ability to self-organize and creates prerequisites for self-development.

The material of the tutorial corresponds to the curriculum of the discipline «Probability Theory and Mathematical Statistics», has close interdisciplinary links with «Higher Mathematics», and is the basis for studying «Statistics». It is recommended to use it in conjunction with the textbook «Probability Theory and Mathematical Statistics».

TOPIC 1. BASIC CONCEPTS OF THE THEORY OF PROBABILITY. ELEMENTS OF COMBINATORICS

PLAN

1. Basic Combinatorial Formulas.
2. Algebra of events. Dependent and independent events. The opposite event and its probability.
3. Classical definition of probability.

Key notions, definitions and categories to be studied: *The acquaintance with the basic concepts of probability theory.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

How many different permutations can be formed from all the letters of the word *mathematics*?

Solution.

There are repeating letters in the word *mathematics*. Let's write these letters and the number of their repetitions:

Mathematics

m -2 times;

a - 2 times;

t – 2 times.

Thus, to answer the question of the task, we must use the repetitive permutation formula

$$P_n(n_1, n_2, \dots n_k) = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$$

There are 11 letters in the word *mathematics*. So, n=11. The number of repetitions of the letter m is equal to 2, that is $n_1 = 2$. Analogically, the number of repetitions of the letter a is equal to 2, that is $n_2 = 2$, and the number of repetitions of the letter t is equal to 2, that is $n_3 = 2$.

We get the next result:

$$P_{11}(2,2,2) = \frac{11!}{2!\cdot 2!\cdot 2!} = \frac{7!\cdot 8\cdot 9\cdot 10\cdot 11}{8} = 4\ 989\ 600.$$

Task № 2

How many different words can be formed from all the letters of the word *person*? How many of them are such that the letters «*p*» and «*e*» are next to each other?

Solution.

1) There are 6 letters in the word *person*. So, *n=6*. We use the permutation formula:

$$P_6 = 6! = 720$$

2) If the letters *p* and *e* are written side by side, then we will consider them as one letter. However, the letters p and e can be next to each other when they are written in this way: «*p e*» and when they are written in a different order: «e p». So, we get the next result:

$$2\cdot P_5 = 2\cdot 5! = 240$$

Task № 3

How many five-digit numbers are there? How many of them start with the number 2 and end with the number 4? How many of them don't contain the number 5? Which are divisible by 5?

Solution.

1) How many ways can we choose the first digit? The number of such ways is equal to 9 (there are 10 digits in total, but we cannot use 0 to write the first digit). How many ways can we choose he second, third, fourth, fifth digit? Obviously, the number of such methods is equal to 10. We can write:

$$k = 9\cdot 10\cdot 10\cdot 10\cdot 10 = 90\ 000$$

2) We know the first and last digits in the number, that is, we can choose them in one way. We choose the second, third, fourth digits in 10 ways. So, we can write:

$$k = 1\cdot 10\cdot 10\cdot 10\cdot 1 = 1\ 000$$

3) If the number 5 is not used in the recording of the number, then we have not 10, but 9 digits. We cannot use 0 to write the first digit. This means that we can choose the first digit in 8 ways. We can choose the second, third, fourth and fifth digits in 9 ways. So, we get the next result:

$$k = 8 \cdot 9 \cdot 9 \cdot 9 \cdot 9 = 52\,488$$

4) If a number is divisible by 5, then it ends with either 0 or 5. Thus, we can choose the last digit in two ways. We can choose the first number in nine ways, and the rest of the numbers we can choose in ten ways. Thus, we can write:

$$k = 9 \cdot 10 \cdot 10 \cdot 10 \cdot 2 = 18\,000$$

Task № 4

How many five-digit numbers can be made from digits: 1, 2, 4, 6, 7, 8, if no digit is used more than once? How many even numbers will be? How many odd numbers will be?

Solution.

1) The numbers in the number record are not repeated. We can choose the first digit in six ways, the second digit - in five ways, we can choose the third digit in four ways, the fourth digit -in three ways, and the fifth digit - in two ways. Thus, we can write

$$k = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$$

2) The number will be even if the last digit is one of the following digits:
2, 4, 6, 8. Thus, we obtain

$$k = 4 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 480$$

3) The number will be odd if the last digit is one of the following digits:
1, 7. Thus, we obtain

$$k = 2 \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) = 240$$

Task № 5

There are 20 balls in the box, among them 8 are red, and the rest are white. Find the probability that: 1) the ball drawn (pulled out) at random will be white; 2) the ball drawn (pulled out) at random will be red.

Solution.

1) Let random event A – the ball drawn at random will be white. The total number of the balls is equal to 20, so the number of the possible outcomes of the trial n=20. Since the number of red balls is 8, the number of white balls is 12, because the total number of balls is 20. So the number of

elementary outcomes of the trial, favorable to event A m=12. Using the classical definition of the probability we can write

$$P=m/n=12/20=3/5.$$

2) Let random event A – the ball drawn at random will be red. The total number of the balls is equal to 20, so the number of the possible outcomes of the trial n=20. The number of red balls is equal to 8, so the number of elementary outcomes of the trial, favorable to event A is equal to m=8. Using the classical definition of the probability we can write

$$P=m/n=8/20=2/5.$$

Task № 6

The dice are tossed once. Find the probability of getting 5 points.

Solution.

Let random event A – tossing the dice will get result in 5 points. On tossing the dice, 1, 2, ..., 6 points can fall, so the total number of outcomes of the trial n=6.

Five points can be tossed only once (we toss the dice exactly once). So, the number of elementary outcomes of the trial, favorable to event A m=1. Using the classical definition of the probability we can write

$$P=m/n=1/6.$$

Task № 7

The student has learned 10 questions out of 15. Find the probability that:

1) he will answer the question; 2) he won’t answer the question.

Solution.

1) Let random event A – the student will answer the question. The total number of the questions is equal to 15. So, we can write n=15. The student has learned only 10 questions, so we can write m=10. Using the classical definition of the probability we can write

$$P=10/15=2/3.$$

2) Let random event A – the student won’t answer the question.
The total number of the questions is equal to 15. So, we can write n=15.
The student has learned 10 questions out of 15, so he don’t know 5 questions and we can write m=5. Using the classical definition of the probability we can write

$$P=5/15=1/3.$$

Task № 8

The box contains 12 details made by the first worker and 6 details made by the second worker. Find the probability that a randomly taken detail will be produced by the first worker.

Solution.

Let random event A - a randomly taken detail will be produced by the first worker. The total number of the details is equal to 18, so the number of the possible outcomes of the trial n=18. And only 12 details are produced by the first worker, so the number of outcomes, favorable to A is equal to m=12. Using the classical definition of the probability we can write

$$P=12/18=2/3.$$

Task № 9

Choose one card from the playing deck (36 pieces). Find the probability that: 1) it will be a card of spades; 2) it will be a lady; 3) it will be the ace of hearts.

Solution.

1) Let random event A - we will take a card of spades. The total number of the cards is equal to 36, so the number of the possible outcomes of the trial n=36. There are 9 cards of spades in the deck. So, the number of outcomes, favorable to A is equal to m=9. The sought probability is equal to:

$$P=9/36=1/4.$$

2) Let random event A - we will take a lady. There are 4 ladies in the deck. So, the number of outcomes, favorable to A is equal to m=4. The sought probability is equal to:

$$P=4/36=1/9.$$

3) Let random event A - we will take an ace of hearts. There are 1 ace of hearts in the deck. So, the number of outcomes, favorable to A is equal to m=1. The sought probability is equal to

$$P=1/36=1.$$

Task № 10

Two dice are tossed. Find the probability that: 1) the sum of the points will be equal to 5; 2) the sum of points will be equal to 14; 3) the sum of points will be less than 3; 4) the sum of points will be more than 9.

Solution.

When we toss the first dice, we can get 1,2,3,4,5,6 points. When we toss the second dice we can also get 1,2,3,4,5,6. Let's write down the possible outcomes of the trial (tossing the dice):

(1,1) (1,2) (1,3) … (1,6) {6 combinations}
(2,1) (2,2) (2,3)…(2,6) {6 combinations}
…
(6,1) (6,2) (6,3) … (6,6) {6 combinations}

So, we get 36 possible outcomes of the trial, so n=36. Let's write down the random event A.

1) Let A - the sum of the points will be equal to 5. Write outcomes of the trial favorable to A: (1,4) (2,3) (3,2) (4,1) {4 combinations}. That is m=4. Using the classical definition of the probability we can write

$$P=4/36=1/9.$$

2) Let A - the sum of the points will be equal to 14. The number of outcomes of the trial favorable to A is equal to zero, because it is impossible to get a total of 14 points when we toss up 2 coins. So, random event A – is an impossible event and its probability:

$$P=0.$$

3) Let A - the sum of the points will be less than 3. Write outcomes of the trial favorable to A: (1,1) {1 combination}. That is m=1. Using the classical definition of the probability we can write:

$$P=1/36.$$

4) Let A - the sum of the points will be more than 9. Write outcomes of the trial favorable to A: (4,6) (5,5) (5,6) (6,4) (6,5) (6,6). So, m=6. We get the next result:

$$P=6/36=1/6.$$

Task № 11

Two dice are tossed. Find the probability that the sum of points that fell is not more than ten.

Solution.

When we toss the first dice, we can get 1,2,3,4,5,6 points. When we toss the second dice we can also get 1,2,3,4,5,6. Let's write down the possible outcomes of the trial (tossing the dice):

(1,1) (1,2) (1,3) … (1,6) {6 combinations}
(2,1) (2,2) (2,3)…(2,6) {6 combinations}
…
(6,1) (6,2) (6,3) … (6,6) {6 combinations}

So, we get 36 possible outcomes of the trial, so n=36. Let's write down the random event A. Let A - the sum of points that fell is not more than ten. Write outcomes of the trial favorable to A:

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6) {6 combinations}
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6) {6 combinations}
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6) {6 combinations}
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6) {6 combinations}
(5,1) (5,2) (5,3) (5,4) (5,5) {5 combinations}
(6,1) (6,2) (6,3) (6,4) {4 combinations}

We get that m=4. The sought probability is equal to

P=4/36=1/9

Task № 12

There are 20 products in the batch, 5 of them have a defect. Three products are chosen at random. Find the probability that at least one of the selected products will be defective.

Solution.

Let random event A - at least one of the selected products will be defective. Consider the opposite event $\bar{A} -$ all selected products won't have any defect.

How many ways can we choose three products out of 20 products? The number of these ways is calculated by the formula (the number of combinations of 20 elements by 3 elements)

$$n = C_{20}^{3} = \frac{20!}{3! \cdot (20-3)!} = \frac{17! \cdot 18 \cdot 19 \cdot 20}{6 \cdot 17!} = 1140$$

How many ways can we choose three products out of the fifteen products that don't have defect? The number of these ways is calculated by the formula (the number of combinations of 15 elements by 3 elements)

$$m = C_{15}^{3} = \frac{15!}{3! \cdot (15-3)!} = \frac{12! \cdot 13 \cdot 14 \cdot 15}{6 \cdot 12!} = 455$$

Using the classical definition of the probability we can write

$$P(\bar{A}\,) = 455/1140 \approx 0{,}4$$

The

$$P(A) = 1 - P(\bar{A}\,) = 1 - 0{,}4 = 0{,}6$$

Task № 13

The student knows 20 of the program’s 25 questions. Calculate the probability that the student knows the three questions suggested by the teacher.

Solution.

How many ways can we choose three questions out of 25 questions? The number of these ways is calculated by the formula (the number of combinations of 25 elements by 3 elements)

$$n = C_{25}^{3} = \frac{25!}{3! \cdot (25-3)!} = \frac{22! \cdot 23 \cdot 24 \cdot 25}{6 \cdot 22!} = 2300$$

How many ways can we choose three of the twenty questions that a student knows? The number of these ways is calculated by the formula (the number of combinations of 20 elements by 3 elements)

$$m = C_{20}^{3} = \frac{20!}{3! \cdot (20-3)!} = \frac{17! \cdot 18 \cdot 19 \cdot 20}{6 \cdot 17!} = 1140$$

Using the classical definition of the probability we can write

$$P = m/n = 1140/2300 \approx 0{,}5$$

Task № 14

From the nine cards with the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, three are chosen at random. What is the probability of getting the number 327?

Solution.

Let random event A - getting the number 327. The number of ways in which three elements can be selected from nine elements, taking into account the fact that the order of the selected elements matters, can be calculated by the formula

$$n = \mathrm{A}_{9}^{3} = \frac{9!}{(9-3)!} = \frac{6! \cdot 7 \cdot 8 \cdot 9}{6!} = 504$$

There is only one way to get the number 327. So, we can write

$$m = 1$$

The probability of the random event A we can calculated by the formula

$$P(A) = \frac{m}{n} = \frac{1}{504}$$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. There are 12 cards with the letters E, B, N, I, G, R, Y, T, H, D, A, M. Take 8 cards in sequence. What is the probability that you get the word *birthday*?
Answer: $P \approx 0$

2. A five-digit number is chosen at random. Find the probability that the number is a multiple of ten.
Answer: $P = 0{,}1$
3. There are 15 balls in a box. Seven of them are white, the rest are black. What is the probability of taking the white ball? What is the probability of taking the black ball? What is the probability of taking two white balls (the taken ball is not returned to the basket)?
Answer: 1) $P = 7/15$*; 2)* $P = 8/15$*; 3)* $P = 1/5$

4. The student learned 9 questions out of 35 in probability theory. If the student answers at least one of the three questions suggested by the teacher, he will pass the exam. What is the probability that the student will pass the exam.
Answer: $P = 0{,}6$

5. The subscriber dialed the phone number and forgot the last two digits. Just remembering that these numbers are different, he dialed them at random. What is the probability that the number is dialed correctly?
Answer: $P = 1/90$

SELF-TEST QUESTIONS

1. How many three-digit numbers can be formed from the digits 4,5,6 if all digits of the formed number must be different:

a. 6
b. 2
c. 1

2. How many ways can you take 2 details from a box containing 10 details?

a. 15
b. 30
c. 45

3. How many different permutations can be formed from all the letters of the word «joke»?

a. 16
b. 24
c. 28

4. There are 10 green balls in a box. What is the probability of taking the green ball?

a. 1
b. 1/4
c. 0

5. The dice are tossed once. Find the probability of getting 5 points.

a. 1/6
b. 1/4
c. 1/2

LITERATURE FOR SELF-STUDY

1. Theodore J. Faticoni Combinatorics: an introduction. First edition. Wiley. 2013. 328 p.
2. Joseph. K. Blitzstein, J. Hwang Introduction to Probability. Second edition. Chapman and Hall/CRC. 2019. 634 p.

TOPIC 2. THEOREMS OF ADDITION AND MULTIPLICATION OF PROBABILITIES

PLAN

1. The theorem of addition of probabilities.
2. The theorem of multiplication of probabilities.
3. Corollaries from the theorems of addition and multiplication of probabilities.

Key notions, definitions and categories to be studied: Key notions, definitions and categories to be studied: *The concept of compatible and incompatible random events. The probability of a random event that can occur if another random event occurs (conditional probability). Basic theorems of probability theory.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

The box contains 10 pencils: 5 red, 2 blue, 3 yellow. Find the probability that: 1) the randomly chosen pencil will be red or blue; 2) the chosen pencil will be yellow or red.

Solution.

1) Let random event A – the randomly chosen pencil will be red or blue. Consider random events A1 – the randomly chosen pencil will be red; A2 – the randomly chosen pencil will be blue. Find the probabilities

$$P(A1) = \frac{5}{10} = \frac{1}{2}$$

$$P(A2) = \frac{2}{10} = \frac{1}{5}$$

Random events A1, A2 are pairwise incompatible, so we can write

$$P(A) = P(A1 + A2) = P(A1) + P(A2) = \frac{1}{2} + \frac{1}{5} = \frac{7}{10} = 0{,}7$$

2) Let random event A – the randomly chosen pencil will be yellow or red. Consider random events A1 – the randomly chosen pencil will be yellow; A2 – the randomly chosen pencil will be red. Find the probabilities

$$P(A1) = \frac{3}{10} = 0{,}3$$

$$P(A2) = \frac{5}{10} = \frac{1}{2} = 0{,}5$$

Random events A1, A2 are incompatible, so we can write

$$P(A) = P(A1 + A2) = P(A1) + P(A2) = 0{,}3 + 0{,}5 = 0{,}8$$

Task № 2

The probability that the first basketball player will hit the ring is 0.8, the probability that the second basketball player will hit the ring is 0.3. Find the probability that when each basketball player makes one throw, then: 1) only the first basketball player will hit; 2) only one basketball player will hit; 3) both basketball players will hit; 4) both basketball players won't be in the basket.

Solution.

1) Let random event A – only the first basketball player will hit; A1 – the first basketball player will hit; A2 – the second basketball player will hit. Then we can write

$$A = A1 \cdot \overline{A2}$$

Write the probabilities of the random events $A1$, $\overline{A2}$.

$$P(A1) = 0{,}8 \quad P(\overline{A2}) = 1 - P(A2) = 1 - 0{,}3 = 0{,}7$$

Events A1, $\overline{A2}$– independent events, so the probability of the random event A is

$$P(A) = P(A1 \cdot \overline{A2}) = P(A1) \cdot P(\overline{A2}) = 0{,}8 \cdot 0{,}7 = 0{,}56$$

2) Let random event A – only one basketball player will hit; A1 – the first basketball player will hit; A2 – the second basketball player will hit. Then we can write:

$$A = A1 \cdot \overline{A2} + \overline{A1} \cdot A2$$

Write the probabilities of the random events $A1$, $A2$, $\overline{A1}$, $\overline{A2}$.

$$P(A1) = 0{,}8 \quad P(A2) = 0{,}3 \quad P(\overline{A1}) = 1 - P(A1) = 1 - 0{,}8 = 0{,}2$$

$$P(\overline{A2}) = 1 - P(A2) = 1 - 0{,}3 = 0{,}7$$

Events A1, $A2$ are independent. Complex events $A1 \cdot \overline{A2}$ and $A1 \cdot \overline{A2}$ are incompatible. So, the probability of the random event A is

$$P(A) = P(A1 \cdot \overline{A2} + \overline{A1} \cdot A2) = P(A1 \cdot \overline{A2}) + P(\overline{A1} \cdot A2) = P(A1) \cdot P(\overline{A2}) + P(\overline{A1}) \cdot P(A2) = 0{,}8 \cdot 0{,}7 + 0{,}2 \cdot 0{,}3 = 0{,}62$$

3) Let random event A – both basketball players will hit; A1 – the first basketball player will hit; A2 – the second basketball player will hit. Then we can write:

$$A = A1 \cdot A2$$

Write the probabilities of the random events $A1$, $A2$.

$$P(A1) = 0{,}8 \quad P(A2) = 0{,}3$$

Events A1, $A2$– independent events, so the probability of the random event A is

$$P(A) = P(A1 \cdot A2) = P(A1) \cdot P(A2) = 0{,}8 \cdot 0{,}3 = 0{,}24$$

4) Let random event A - both basketball players won't hit in the basket;
A1 – the first basketball player won't hit; A2 – the second basketball player won't hit. Then we can write

$$A = \overline{A1} \cdot \overline{A2}$$

Write the probabilities of the random events $\overline{A1}$, $\overline{A2}$

$$P(\overline{A1}) = 1 - 0{,}8 = 0{,}2 \quad P(\overline{A2}) = 1 - 0{,}3 = 0{,}7$$

Events $\overline{A1}$, $\overline{A2}$– independent events, so the probability of the random event A is

$$P(A) = P(\overline{A1} \cdot \overline{A2}) = P(\overline{A1}) \cdot P(\overline{A2}) = 0{,}2 \cdot 0{,}7 = 0{,}14$$

Task № 3

Three shooters fired one shot each. The probability of hitting each of them is 0.3, 0.4; 0.6 respectively. Find the probability that there will be only one hit on the target.

Solution.

Let random event A – there will be only one hit on the target; A1 – the first shooter hits the target; A2 – the second shooter hits the target; A3 – the third shooter hits the target. Events A1, A2, A3 – independent events. By the condition of the task

$$P(A1) = 0{,}3 \quad P(A2) = 0{,}4 \quad P(A3) = 0{,}6$$

Find probabilities:

$$P(\overline{A1}) = 1 - P(A1) = 1 - 0{,}3 = 0{,}7$$

$$P(\overline{A2}) = 1 - P(A2) = 1 - 0{,}4 = 0{,}6$$

$$P(\overline{A3}) = 1 - P(A3) = 1 - 0{,}6 = 0{,}4$$

Consider complex events

$$A1 \cdot \overline{A2} \cdot \overline{A3}, \quad \overline{A1} \cdot A2 \cdot \overline{A3}, \quad \overline{A1} \cdot \overline{A2} \cdot A3$$

We can write

$$A = A1 \cdot \overline{A2} \cdot \overline{A3} + \overline{A1} \cdot A2 \cdot \overline{A3} + \overline{A1} \cdot \overline{A2} \cdot A3$$

These events are pairwise incompatible. So, we can write:

$$\begin{aligned} P(A) &= P(A1 \cdot \overline{A2} \cdot \overline{A3} + \overline{A1} \cdot A2 \cdot \overline{A3} + \overline{A1} \cdot \overline{A2} \cdot A3) \\ &= P(A1 \cdot \overline{A2} \cdot \overline{A3}) + P(\overline{A1} \cdot A2 \cdot \overline{A3}) + P(\overline{A1} \cdot \overline{A2} \cdot A3) = \\ &= P(A1) \cdot P(\overline{A2}) \cdot P(\overline{A3}) + P(\overline{A1}) \cdot P(A2) \cdot P(\overline{A3}) + P(\overline{A1}) \\ &\cdot P(\overline{A2}) \cdot P(A3) \end{aligned}$$

$$\begin{aligned} P(A) &= 0{,}3 \cdot 0{,}6 \cdot 0{,}4 + 0{,}7 \cdot 0{,}4 \cdot 0{,}4 + 0{,}7 \cdot 0{,}6 \cdot 0{,}6 \\ &= 0{,}072 + 0{,}112 + 0{,}252 = 0{,}436 \end{aligned}$$

Task № 4

On five cards the numbers 8, 5, 3, 9, 2 are written. Three cards are sequentially taken at random and laid out in a row. Find the probability that you get the number 385.

Solution.

Let random event A - getting the number 385. Consider events A1 – the first we take the card with the number 3; A2 – the second we will take a card with a number 8; A3 – the third we will take a card with a number 5. We can write:

$$A = A_1 \cdot A_2 \cdot A_3$$

Events A1, A2, A3 are compatible events, so the probability of the random event A is

$$P(A) = P(A_1 \cdot A_2 \cdot A_3) = P(A_1) \cdot P_{A_1}(A_2) \cdot P_{A_1 \cdot A_2}(A_3)$$

$$P(A_1) = \frac{1}{5}$$

$$one\ card\ was\ taken, four\ cards\ left$$

$$P_{A_1}(A_2) = \frac{1}{4}$$

$$(one\ card\ was\ taken, three\ cards\ left)$$

$$P_{A_1 \cdot A_2}(A_3) = \frac{1}{3}$$

So, we get the next result

$$P(A) = \frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{60}$$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. The average number of rainy days in August is equal to four. Find the probability that the first three days of the month will be rainy.

Answer: $P \approx 0{,}001$

2. The probability that a student will answer the first question is 70 %, the probability that he will answer the second question is 60 %. Find the probability that: 1) the student will not answer the first question, and will answer the second; 2) the student will not answer the questions; 3) the student will answer at least one question.

Answer: 1) $P = 0{,}42$*; 2)* $P = 0{,}12$*; 3)* $P = 0{,}88$

3. Three basketball players must make one throw of the ball. The probability of the ball hitting the basket for the first, second and third basketball players is 0.9; 0.8; 0.7. Find the probability that at least one basketball player will successfully throw the ball.

Answer: $P = 0{,}994$

SELF-TEST QUESTIONS

1. The student has learned 10 questions out of 15. The probability that he won't answer the question is
   a. 1/3
   b. 1
   c. 0

2. The box contains 12 details made by the first worker and 6 details made by the second worker. The probability that a randomly taken detail will be produced by the first worker is
   a. 1/3
   b. 2/3
   c. 1/2

3. There are 20 products in the batch, 5 of them have a defect. Find the probability that the randomly selected product will be defective.
a. 1/4
b. 1/3
c. 1/2

4. Two shooters fired one shot each. The probability of hitting the target by the first shooter is 0.6, for the second shooter this probability is equal to 0,8. What is the probability that both shooters hit the target?
a. 0,6
b. 0,48
c. 0,8

5. The box contains 10 pencils: 5 red, 2 blue, 3 yellow. Find the probability that the randomly chosen pencil will be red or blue.
a. 0,6
b. 0,7
c. 0,5

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2. Prasanna Sahoo Probability and mathematical statistics: Textbook. USA: Department of Mathematics of the University of Louisville, 2013. 712 p.
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TOPIC 3. FORMULA OF COMPLETE PROBABILITY. BAYES' FORMULA

PLAN

1. Formula of complete probability.
2. Bayes' formula.

Key notions, definitions and categories to be studied: *Formula of complete probability. Bayes' formula. Features of the use of the formula of complete probability and Bayes' formula in solving probabilistic tasks.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

A white ball is put into the basket containing two balls. After that, one ball is taken at random from it. Find the probability that a white ball will be taken if all possible assumptions about the initial composition of the balls (by color) are equally possible.

Solution.

Let random event A - a white ball will be taken. Let put forward hypotheses about the initial composition of the balls:

B1- There were no white balls in the basket.
B2 - There was one white ball in the basket.
B3 - There were two white balls in the basket.

By the condition of the task, all possible assumptions about the initial composition of the balls (by color) are equally possible. Find the probabilities of hypotheses:

$$P(B1) = P(B2) = P(B3) = \frac{1}{3}$$

Find condition probabilities of the random event A:

$$P_{B1}(A) = \frac{1}{3}$$

{the probability that a white ball will be taken, calculated under assumption that there were no white balls in the basket initially}

$$P_{B2}(A) = \frac{2}{3}$$

{the probability that a white ball will be taken, calculated under assumption that there were one white ball in the basket initially}

$$P_{B3}(A) = \frac{3}{3} = 1$$

{the probability that a white ball will be taken, calculated under assumption that there were two white balls in the basket initially}

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A) = \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = \frac{2}{3}$$

Task № 2

There are three identical baskets. The first basket contains 4 white and 3 black balls. The second basket contains 2 white and 3 black balls. The third basket contains only 2 white balls. A ball was taken from a randomly selected basket. Find the probability that this ball will be white.

Solution.

Let random event A - a white ball will be taken. Put forward hypotheses:

B1- The ball was taken from the first basket.
B2 - The ball was taken from the second basket.
B3 - The ball was taken from the third basket.

By the condition of the task, there are three *identical* baskets. So, events B1, B2, B3 are equally possible. Find the probabilities of hypotheses:

$$P(B1) = P(B2) = P(B3) = \frac{1}{3}$$

Find condition probabilities of the random event A:

$$P_{B1}(A) = \frac{4}{7}$$

$\{the\ probability\ that\ a\ white\ ball\ was\ taken\ from\ the\ first\ basket\ \}$

$$P_{B2}(A) = \frac{2}{5}$$

$$\{the\ probability\ that\ a\ white\ ball\ was\ taken\ from\ the\ second\ basket\ \}$$

$$P_{B3}(A) = \frac{2}{2} = 1$$

$$\{the\ probability\ that\ a\ white\ ball\ was\ taken\ from\ the\ third\ basket\ \}$$

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A) = \frac{1}{3}\left(\frac{4}{7} + \frac{2}{5} + 1\right) = \frac{23}{35}$$

Task № 3

TVs of three factories go on sale. The products of the first factory have 20 % of TVs with a hidden defect, the products of the second factory have 10 % of TVs with a defect, the products of the third factory have 5 % of TVs with a defect. What is the probability of buying a TV without a defect, if the store received 30 % of TVs from the first factory, 20 % – from the second factory, 50 % – from the third factory?

Solution.

Let random event A - buying a TV without a defect. Put forward hypotheses:

B1- The TV was produced by the first factory.
B2 - The TV was produced by the second factory.
B3 - The TV was produced by the third factory.

Find the probabilities of hypotheses:

$$P(B1) = 0{,}3$$

$$P(B2) = 0{,}2$$

$$P(B3) = 0{,}5$$

Find condition probabilities of the random event A:

$$P_{B1}(A) = 1 - 0{,}2 = 0{,}8$$

$$\{the\ probability\ that\ the\ TV\ \ produced\ by\ the\ first\ factory\ has\ no\ defect\}$$

$$P_{B2}(A) = 1 - 0{,}1 = 0{,}9$$

$$\{\ the\ probability\ that\ the\ TV\ \ produced\ by\ the\ second\ factory\ has\ no\ defect\}$$

$$P_{B3}(A) = 1 - 0{,}05 = 0{,}95$$

$$\{the\ probability\ that\ the\ TV\ \ produced\ by\ the\ third\ factory\ has\ no\ defect\}$$

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A) = 0{,}3 \cdot 0{,}8 + 0{,}2 \cdot 0{,}9 + 0{,}5 \cdot 0{,}95 = 0{,}895$$

Task № 4

There are 500 lamps in the batch: 250 are made at the 1st plant, 150 – at the 2nd, 100 – at the 3rd plant. The probability that a randomly selected lamp will be standard for the first plant is 0.97, for the 2nd plant – 0.92, for the 3rd plant – 0.93. What is the probability that a randomly selected lamp will be standard?

Solution.

Let random event A - a randomly selected lamp will be standard. Put forward hypotheses:

B1- A randomly selected lamp is made at the 1st plant.

B2 - A randomly selected lamp is made at the 2nd plant.

B3 - A randomly selected lamp is made at the 3rd plant.

Find the probabilities of hypotheses:

$$P(B1) = \frac{250}{500} = \frac{1}{2} = 0{,}5$$

$$P(B2) = \frac{150}{500} = \frac{3}{10} = 0{,}3$$

$$P(B3) = \frac{100}{500} = 0{,}2$$

Find condition probabilities of the random event A:

$$P_{B1}(A) = 0{,}97$$

$\{the\ probability\ that\ a\ randomly\ selected\ lamp\ will\ be\ standard\ and\ it\ is\ made\ at\ the\ 1st\ plant\}$

$$P_{B2}(A) = 0{,}92$$

$\{the\ probability\ that\ a\ randomly\ selected\ lamp\ will\ be\ standard\ and\ it\ is\ made\ at\ the\ 2nd\ plant\}$

$$P_{B3}(A) = 0{,}93$$

$\{the\ probability\ that\ a\ randomly\ selected\ lamp\ will\ be\ standard\ and\ it\ is\ made\ at\ the\ 3rd\ plant\}$

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A) = 0{,}5 \cdot 0{,}97 + 0{,}3 \cdot 0{,}92 + 0{,}2 \cdot 0{,}93 = 0{,}947$$

Task № 5

The store sells the same type of products from three companies in equal quantities. For products produced by the first company, the probability of detecting a defect is 0.1, for products of the second company this probability is equal to 0.05, and for products of the third company, the probability of detecting a defect is 0.02. The product taken at random turned out to be defective. What is the probability that it is made by the first company?

Solution.

Let random event A - a randomly selected product will be defective. Put forward hypotheses:

B1- A randomly selected product was made by the first company.

B2 - A randomly selected product was made by the second company.

B3 - A randomly selected product was made by the third company.

By the condition of the task, the store sells the same type of products from three companies in equal quantities. Find the probabilities of hypotheses:

$$P(B1) = P(B2) = P(B3) = \frac{1}{3}$$

Find condition probabilities of the random event A:

$$P_{B1}(A) = 0{,}1$$

$\{the\ probability\ that\ a\ randomly\ selected\ product\ will\ be\ defective\ and\ it\ is\ made\ by\ the\ 1st\ company\}$

$$P_{B2}(A) = 0{,}05$$

$\{the\ probability\ that\ a\ randomly\ selected\ product\ will\ be\ defective\ and\ it\ is\ made\ by\ the\ 2nd\ company\}$

$$P_{B3}(A) = 0{,}02$$

$\{the\ probability\ that\ a\ randomly\ selected\ product\ will\ be\ defective\ and\ it\ is\ made\ by\ the\ 3rd\ company\}$

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A)=\frac{1}{3}\cdot(0{,}1+0{,}05+0{,}02)=\frac{1}{3}\cdot 0{,}17\approx 0{,}06$$

Event A has already occurred. We recalculate the probability of the hypothesis B1:

$$P_A(B1)=\frac{P(B1)\cdot P_{B1}(A)}{P(A)}\qquad\{Bayes\ formula\}$$

Thus, we get the next result:

$$P_A(B1)=\frac{\frac{1}{3}\cdot 0{,}1}{\frac{1}{3}\cdot 0{,}17}\approx 0{,}59$$

Task № 6

4 students from the 1st year, 5 students from the 2nd year, 6 students from the 3rd year were selected to participate in the competitions. The probability that the 1st, 2nd and 3rd year student will get into the national team is 0.9; 0.7; 0.8 respectively. Randomly selected student got into the team. Find the probability that the first-year student was selected.

Solution.

Let random event A – a randomly selected student will get into the team. Put forward hypotheses:

B1- the first-year student will be selected.
B2 - the second-year student will be selected.
B3 - the third-year student will be selected.

The total number of the selected students is equal to 4+5+6=15. Four of them are first-year students, five of them are second-year students, and six of the selected students are third-year students. Find the probabilities of hypotheses:

$$P(B1)=\frac{4}{15}$$
$$P(B2)=\frac{5}{15}=\frac{1}{3}$$
$$P(B3)=\frac{6}{15}=\frac{2}{5}$$

Find condition probabilities of the random event A:

$$P_{B1}(A)=0{,}9$$

$$\{the\ probability\ that\ \ the\ first - year\ student\ will\ get\ into\ the\ team\ \}$$

$$P_{B2}(A) = 0{,}7$$

$$\{\ the\ probability\ that\ \ the\ second - year\ student\ will\ get\ into\ the\ team\}$$

$$P_{B3}(A) = 0{,}8$$

$$\{\ the\ probability\ that\ \ the\ third - year\ student\ will\ get\ into\ the\ team\}$$

Calculate the probability of a random event A. We use the formula of the complete probability:

$$P(A) = P(B_1) \cdot P_{B_1}(A) + P(B_2) \cdot P_{B_2}(A) + P(B_3) \cdot P_{B_3}(A)$$

$$P(A) = \frac{4}{15} \cdot 0{,}9 + \frac{5}{15} \cdot 0{,}7 + \frac{6}{15} \cdot 0{,}8 = \frac{1}{15}(3{,}6 + 3{,}5 + 4{,}8) = \frac{1}{15} \cdot 11{,}9 \approx 0{,}79$$

Event A has already occurred. We recalculate the probability of the hypothesis B1:

$$P_A(B1) = \frac{P(B1) \cdot P_{B1}(A)}{P(A)} \qquad \{Bayes\ formula\}$$

Thus, we get the next result:

$$P_A(B1) = \frac{\frac{4}{15} \cdot 0{,}9}{\frac{1}{15} \cdot 11{,}9} = \frac{3{,}6}{11{,}9} \approx 0{,}30$$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. Two machines produce the same products. The productivity of the first machine is twice as much as the productivity of the second machine. The first machine gives 60 % of first-class products, the second machine gives 84 % of first-class products. What is the probability that a randomly selected product will be first class?
Answer: $P = 0{,}68$

2. The number of trucks driving on a highway where a gas station is located is 3: 2 as compared to passenger cars driving on the same highway. The probability that a truck will refuel is 0.1. For a passenger car, the same

probability is 0.2. A car drove up to the gas station. Find the probability that it is a truck.

Answer: $P \approx 0{,}43$

3. The first basket contains 10 balls, 8 of them are white. The second basket contains 20 balls, 4 of them are white. One ball was taken at random from each basket, and then one ball was taken at random from these two balls. Find the probability that a white ball is taken.

Answer: $P = 0{,}5$

SELF-TEST QUESTIONS

1. The probability of an event A, which can occur only if one of the incompatible events B_1, B_2, …, B_n, forming a complete group of events, occurs, is determined by
   a. the Bayes' formula
   b. the Bernoulli's formula
   c. the formula of complete probability

2. Let event A occur only if one of the incompatible events B_1, B_2, …, B_n which form a complete group occurs. If event A has already occurred then probabilities of hypotheses can be overestimated by
   a. the Bernoulli's formula
   b. the Bayes' formula
   c. the formula of complete probability

3. In the formula of complete probability, events B_1, B_2, …, B_n (hypotheses) must satisfy the condition
   a. $P(B_1) + P(B_2) + \cdots + P(B_n) = 0$
   b. $P(B_1) + P(B_2) + \cdots + P(B_n) = 1$
   c. $P(B_1) + P(B_2) + \cdots + P(B_n) = n$

4. An event A can occur if one of the incompatible events B_1, B_2, …, B_n forming a complete group of events, occurs. The probabilities of hypotheses B_1 and B_2 are known: $P(B_1) = 0{,}6$; $P(B_2) = 0{,}3$. Find the probability of hypothesis B_3.
   a. 0,3
   b. 0,2
   c. 0,1

5. A white ball is dropped into a box containing 3 balls, after which one ball is drawn at random. Find the probability that the drawn ball will be white if all possible assumptions about the initial composition of the balls (by color) are equally possible.

   a. $3/8$
   b. $5/8$
   c. $7/8$

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1. V. E. Gmurman, I. I Berenblut Fundamentals of probability theory and mathematical statistics. [Electronic edition]

Access mode: https://www.worldcat.org/title/fundamentals-of-probability-theory-and-mathematical-statistics/oclc/499092704?referer=di&ht=edition

2. Joseph. K. Blitzstein, J. Hwang Introduction to Probability. Second edition. Chapman and Hall/CRC. 2019. 634 p.

3. S.P. Thompson A textbook of higher mathematics: learning calculus in a simple way. Access mode: https://www.amazon.com/TextBook-Higher-Mathematics-Integration-Differentiation-ebook/dp/B074TVTQ32.

TOPIC 4. BERNOULLI'S FORMULA. LOCAL AND INTEGRAL THEOREMS OF LAPLACE

PLAN

1. Repeat trials.
2. Formulas of Bernoulli, Poisson and Muavra-Laplace.

Key notions, definitions and categories to be studied: *The concept of repetitive tests. Independent of some random test event. Features of the use of formulas Bernoulli, Poisson and Muavra -Laplace.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

The dice are tossed 4 times. Find the probability that two points will fall out 3 times.

Solution.

Let random event A - getting two points on a coin toss. The probability that two points will fall out is equal to p=1/6. This probability is the same for every toss of the dice. Thus, a series of trials is carried out, in each of which the probability of the event A is the same and it is equal to 1/6. To solve the problem, we apply the *Bernoulli's formula*

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

In accordance with the task, we can write n=4, p=1/6, q=1-p=5/6. The sought probability is:

$$P_4(3) = C_4^3 \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right) = \frac{4!}{3!\,(4-3)!} \cdot \frac{5}{1296} = \frac{3! \cdot 4}{3! \cdot 1} \cdot \frac{5}{1296} = \frac{5}{324}$$

Task № 2

The coin is tossed five times. Find the probability that the coat of arms will fall out: 1) less than 2 times; 2) at least 2 times.

Solution.

Let random event A - the coat of arms will fall out. The probability that the coat of arms will fall out is equal to p=1/2. This probability is the same for every coin toss. Thus, a series of trials is carried out, in each of which the probability of the event A is the same and is equal to 1/2. To solve the problem, we apply the *Bernoulli's formula*

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

In accordance with the task, we can write n=5, p=1/2, q=1-p=1/2. The sought probability is:

$$1)\ P_5(< 2) = P_5(0) + P_5(1) = C_5^0 \cdot \left(\frac{1}{2}\right)^0 \cdot \left(\frac{1}{2}\right)^5 + C_5^1 \cdot \left(\frac{1}{2}\right)^1 \cdot \left(\frac{1}{2}\right)^4 =$$

$$= \left(\frac{1}{2}\right)^5 \cdot (1+5) = \frac{6}{32} = \frac{3}{16}$$

$P_5(0)$ *–the probability that with five tosses of a coin the coat of arms will not fall out even once.*

$P_5(1)$ – *the probability that with five coin tosses the coat of arms will appear once.*

$$2)\ P_5(\geq 2) = 1 - P_5(< 2) = 1 - \frac{3}{16} = \frac{13}{16}$$

Task № 3

Two equivalent chess players are playing. Which is more likely: to win 2 games out of 4 or 3 games out of 6?

Solution.

Let random event A – to win a game of chess. Two equivalent chess players are playing, so the probability of winning for each is equal to p=1/2. Thus, we have repeated trials, in each of which the probability of the event A is constant and equal to p=1/2. To solve the problem, we apply the *Bernoulli's formula*

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m},$$

where *q=1-p*. We get the next result:

$$P_4(2) = C_4^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^2 = \frac{4!}{2!\,(4-2)!} \cdot \left(\frac{1}{2}\right)^4 = \frac{2! \cdot 3 \cdot 4}{2! \cdot 2} \cdot \frac{1}{16} = \frac{6}{16} = \frac{3}{8}$$

$$P_6(3) = C_6^3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^3 = \frac{6!}{3!\,(6-3)!} \cdot \left(\frac{1}{2}\right)^6 = \frac{3! \cdot 4 \cdot 5 \cdot 6}{3! \cdot 6} \cdot \frac{1}{64} = \frac{5}{16}$$

And $P_4(2) > P_6(3)$, *so it is more likely to win two games out of four.*

Task № 4

The probability of hitting the target with one shot is 0.75. Find the probability that with five shots: 1) the shooter will hit the target 2 times; 2) the shooter will hit the target less than 2 times; 3) the shooter will hit the target more than 3 times; 4) the shooter will hit the target no more than 3 times.

Solution.

Let random event A - the shooter will hit the target. The probability that the shooter will hit the target is equal to p=0,75. This probability is the same for each shot. Thus, a series of trials is carried out, in each of which the probability of the event A is the same and is equal to 0,75. To solve the problem, we apply the *Bernoulli's formula*

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

In accordance with the task, we can write n=5, p=0,75, q=1-p=0,25. The sought probability is:

$$1)\ P_5(2) = C_5^2 \cdot (0{,}75)^2 \cdot (0{,}25)^3 =$$

$$= \frac{5!}{2!\,(5-2)!} \cdot \left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right)^3 = \frac{3! \cdot 4 \cdot 5}{3! \cdot 2} \cdot \frac{9}{1024} = \frac{45}{512} \approx \quad \approx 0{,}088$$

$$2)\ P_5(<2) = P_5(0) + P_5(1) = C_5^0 \cdot (0{,}75)^0 \cdot (0{,}25)^5 + C_5^1 \cdot (0{,}75) \cdot (0{,}25)^4 =$$

$$= \left(\frac{1}{4}\right)^5 + \frac{5!}{1!\,(5-1)!} \cdot \left(\frac{3}{4}\right) \cdot \left(\frac{1}{4}\right)^4 = \frac{1}{1024} + 5 \cdot \frac{3}{1024} = \frac{16}{1024} = \frac{1}{64} \approx 0{,}016$$

$$3)\ P_5(>3) = P_5(4) + P_5(5) = C_5^4 \cdot (0{,}75)^4 \cdot (0{,}25) + (0{,}75)^5 =$$

$$= \frac{5!}{4!\,(5-4)!} \cdot \left(\frac{3}{4}\right)^4 \cdot \left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5 = 5 \cdot \frac{81}{1024} + \frac{243}{1024} = \frac{648}{1024} = \frac{81}{128} \approx 0{,}633$$

$$4)\ P_5(\le 3) = 1 - P_5(>3) = 1 - \frac{81}{128} = \frac{47}{128} \approx 0{,}367$$

Task № 5

One out of eight TVs requires repair during the warranty period. Find the probability that three out of four TVs will need to be repaired.

Solution.

Let random event A- a television set will need to be repaired. One out of eight TVs requires repair during the warranty period. Thus, the probability that a television set will need to be repaired is equal to p=1/8. This probability is constant value. Thus, a series of trials is carried out, in each of which the probability of the event A is the same and it is equal to 1/8. To solve the problem, we apply the *Bernoulli's formula*

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

In accordance with the task, we can write n=4 p=1/8 q=1-p=7/8. The sought probability is

$$P_4(3) = C_4^3 \cdot \left(\frac{1}{8}\right)^3 \cdot \left(\frac{7}{8}\right)^1 = \frac{4!}{3!\,(4-3)!} \cdot \frac{7}{4096} = \frac{3! \cdot 4}{3!} \cdot \frac{7}{4096} = \frac{7}{1024} \approx 0{,}007$$

Task № 6

Six out of 10 students receive a scholarship. Find the probability that out of 100 randomly selected students 70 will receive a scholarship.

Solution.

Let random event A- a randomly selected student will receive a scholarship.

Six out of 10 students receive a scholarship. Thus, the probability that a randomly selected student will receive a scholarship is equal to p=6/10=3/5. Thus, a series of trials is carried out, in each of which the probability of the event A is the same and is equal to 3/5.

We can write n=100, m=70, p=3/5=0,6, q=1-p=0,4. The number of trials is large enough, while the probability of event A is also not small. To solve the problem, we apply the *Laplace formula*

$$P_n(m) \approx \frac{1}{\sqrt{npq}}\varphi(x_m), \qquad x_m = \frac{m - np}{\sqrt{npq}}$$

Find

$$x_m = \frac{70 - 100 \cdot 0{,}6}{\sqrt{100 \cdot 0{,}6 \cdot 0{,}4}} = \frac{10}{\sqrt{24}} \approx 2{,}04;$$

$$\varphi(x_m) = 0{,}0498.$$

The sought probability is

$$P_{100}(70) \approx \frac{1}{\sqrt{24}} \cdot 0{,}0498 \approx 0{,}01$$

Task № 7

The probability of hitting the target with one shot is 0.8. Find the probability that there will be 310 hits at 400 shots.

Solution.

Let random event A- hitting the target. In accordance with the task, we can write n=400, m=310, p=0,8, q=1-p=0,2. The number of trials is large enough, while the probability of event A is also not small. To solve the problem, we apply the *Laplace formula*

$$P_n(m) \approx \frac{1}{\sqrt{npq}}\varphi(x_m), \qquad x_m = \frac{m-np}{\sqrt{npq}}$$

Find

$$x_m = \frac{310-400\cdot 0{,}8}{\sqrt{400\cdot 0{,}8\cdot 0{,}2}} = -\frac{10}{8} = -1{,}25;$$

$$\varphi(x_m) = 0{,}1826.$$

The sought probability is

$$P_{400}(310) \approx \frac{1}{8}\cdot 0{,}1826 \approx 0{,}023$$

Task № 8

The probability of having a boy is 0.51. Find the probability that there will be 50 boys among 100 newborns.

Solution.

Let random event A- a boy will be born. In accordance with the task, we can write n=100, m=50, p=0,51, q=1-p=0,49. The number of trials is large enough, while the probability of event A is also not small. To solve the problem, we apply the *Laplace formula*

$$P_n(m) \approx \frac{1}{\sqrt{npq}}\varphi(x_m), \qquad x_m = \frac{m-np}{\sqrt{npq}}$$

Find

$$x_m = \frac{50-100\cdot 0{,}51}{\sqrt{100\cdot 0{,}51\cdot 0{,}49}} \approx -0{,}20;$$

$$\varphi(x_m) = 0{,}3910$$

The sought probability is

$$P_{100}(50) \approx \frac{1}{4{,}999}\cdot 0{,}3910 \approx 0{,}078$$

Task № 9

Find the probability that event A will occur exactly 70 times in 243 trials, if the probability of an event in each trial is 0.25.

Solution.

In accordance with the task, we can write n=243, m=70, p=0,25, q=1, p=0,75. The number of trials is large enough, while the probability of event A is also not small. To solve the problem, we apply the *Laplace formula*

$$P_n(m) \approx \frac{1}{\sqrt{npq}}\varphi(x_m), \qquad x_m = \frac{m-np}{\sqrt{npq}}$$

Find

$$x_m = \frac{70 - 243 \cdot 0{,}25}{\sqrt{243 \cdot 0{,}25 \cdot 0{,}75}} = \frac{9{,}25}{6{,}75} \approx 0{,}37$$

$$\varphi(x_m) = 0{,}1561$$

The sought probability is

$$P_{243}(70) \approx \frac{1}{6{,}75} \cdot 0{,}1561 \approx 0{,}023$$

Task № 10

The average number of taxi orders arriving at the control room in one minute is three. Find the probability that in 2 minutes there will be: 1) 4 calls; 2) less than 4 calls; 3) at least 4 calls.

Solution.

Let A − a taxi is ordered. In accordance with the task, we can write $\lambda = 3$ (*the average number of occurrences of the event A per unit time*), $t = 2$, $m = 4$. To solve the problem, we write the *Poisson's formula* in the form:

$$P_t(m) = \frac{(\lambda t)^m}{m!} e^{-\lambda t}$$

1) The sought probability is

$$P_2(4) = \frac{6^4}{4!} \cdot 0{,}0024 \approx 0{,}134$$

2) The sought probability is

$$P_2(<4) = P_2(0) + P_2(1) + P_2(2) + P_2(3) =$$
$$= e^{-6} + \frac{6}{1!} \cdot e^{-6} + \frac{6^2}{2!} \cdot e^{-6} + \frac{6^3}{3!} \cdot e^{-6} = e^{-6}(1 + 6 + 18 + 36)$$
$$\approx\approx 0{,}151$$

3) $P_2(\geq 4) = 1 - P_2(<4) = 1 - 0{,}151 = 0{,}849$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. The probability of hitting a target for a sniper is 0.8. Find the probability that having made 900 shots, the sniper will hit the target: 1) from 700 to 800 times;
2) not less than 750 times; 3) less than 750 times.
Answer: 1) $P_{900}(700; 800) = 0{,}9525$; *2)* $P_{900}(750; 900) = 0{,}0062$; *3)* $P_{900}(0; 749) = 0{,}9938$.

2. The probability of produce a defective detail is 0.3 %. Find the probability that out of 1000 produced details, two will be defective.
Answer: $P_{1000}(2) \approx 0{,}224$

3. The average number of calls of ambulance in three minutes is equal to 6. Find the probability that in one minute there will be: 1) three calls; 2) less than three calls; 3) at least one call.
Answer: 1) $P_1(3) \approx 0{,}1804$; *2)* $P_1(< 3) \approx 0{,}6767$; *3)* $P_1(\geq 1) \approx 0{,}8647$

SELF-TEST QUESTIONS

1. To use Bernoulli's formula it is necessary that
   a. the trials were independent of event A
   b. the trials were dependent of event A
   c. another answer

2. The Bernoulli's formula is used for
   a. the large number of trials (n>20)
   b. the small number of trials (n<20)
   c. the number of trials doesn't matter

3. It is advisable to use the Local theorem of Laplace when it is required to calculate the probability of an event occurring exactly k times in n trials, while
   a. the number of trials is rather small (n>20)
   b. the number of is large (n < 20)
   c. the number of trials does not matter

4. The probability that event A will occur in n independent trials from k_1 to k_2 times can be approximately found using

a. Local theorem of Laplace
b. Integral theorem of Laplace
c. Bernoulli's formula

5. A dice is tossed 600 times. What is the probability that one point will fall out exactly 110 times?
a. $\approx 0{,}07$
b. $\approx 0{,}05$
c. $\approx 0{,}02$

LITERATURE FOR SELF-STUDY

1. V. E. Gmurman, I. I Berenblut Fundamentals of probability theory and mathematical statistics. [Electronic edition]
Access mode: https://www.worldcat.org/title/fundamentals-of-probability-theory-and-mathematical-statistics/oclc/499092704?referer=di&ht=edition
2. Joseph. K. Blitzstein, J. Hwang Introduction to Probability. Second edition. Chapman and Hall/CRC. 2019. 634 p.
3. S.P. Thompson A textbook of higher mathematics: learning calculus in a simple way. Access mode: https://www.amazon.com/TextBook-Higher-Mathematics-Integration-Differentiation-ebook/dp/B074TVTQ32.

TOPIC 5. DISCRETE AND CONTINUOUS RANDOM VARIABLES. INTEGRAL DISTRIBUTION FUNCTION. DIFFERENTIAL DISTRIBUTION FUNCTION

PLAN

1. Discrete and continuous random variables. Basic concepts.
2. Integral and differential distribution functions.

Key notions, definitions and categories to be studied: *The concept of the discrete and* continuous *random variables, integral and differential distribution functions.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

The device consists of three independently operating elements. The probability of failure of each element in one experiment is 0.1. Draw up the distribution law for the number of failed elements in one experiment.

Solution.

A discrete random variable X (the number of failed elements in one experiment) has the following possible values:

x = 0 (none of the elements failed)

x = 1 (one element failed)

x = 2 (two elements failed)

x = 3 (three elements failed)

Failures of elements are independent of each other, the probabilities of failure of each element are equal to each other, so we can apply the Bernoulli formula

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

By the condition of the task n=3; p=0,1; q=1-p=0,9. So, we get the result:

$P_3(0) = q^3 = (0{,}9)^3 = 0{,}729$

$P_3(1) = C_3^1 \cdot p \quad \cdot q^2 = \dfrac{3!}{1! \cdot (3-1)!} \cdot 0{,}1 \cdot (0{,}9)^2 = 3 \cdot 0{,}1 \cdot (0{,}9)^2 =$
$= 0{,}243$

$P_3(2) = C_3^2 \cdot p^2 \cdot q \quad = \dfrac{3!}{2! \cdot (3-2)!} \cdot (0{,}1)^2 \cdot 0{,}9 = 3 \cdot (0{,}1)^2 \cdot 0{,}9 =$
$= 0{,}027$

$P_3(3) = p^3 = (0{,}1)^3 = 0{,}001$

Let's check the values of probabilities:

$$0{,}729 + 0{,}243 + 0{,}027 + 0{,}001 = 1$$

Write the sought binomial distribution law of the random variable X:

X	0	1	2	3
p	0,729	0,243	0,027	0,001

Task № 2

Write the binomial law for the distribution of a discrete random variable
X - the number of times the coat of arms appears in two tosses of the coin.

Solution.

A discrete random variable X (the number of times the coat of arms appears in two tosses of the coin) has the following possible values:

x = 0 (the coat of arms never fell out)
x = 1 (the coat of arms fell out 1 time)
x = 2 (the coat of arms fell out 2 times)

Let's find the probabilities of possible values of X using the Bernoulli's formula

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

By the condition of the task n=2; p=0,5; q=1-p=0,5. So, we get the result:

$P_2(0) = q^2 = (0{,}5)^2 = 0{,}25$

$P_2(1) = C_2^1 \cdot p \quad \cdot q \quad = \dfrac{2!}{1! \cdot (2-1)!} \cdot 0{,}5 \cdot 0{,}5 = 2 \cdot 0{,}25 = 0{,}5$

$P_2(2) = p^2 = (0{,}5)^3 = 0{,}25$

Let's check the values of probabilities:

$$0{,}25 + 0{,}5 + 0{,}25 = 1$$

Let's write the sought binomial distribution law of the random variable X:

X	0	1	2
p	0,25	0,5	0,25

Task № 3

The shooter hits a target with the probability of 0,6. Find the distribution law of a random variable X – the number of hits, if the shooter fires three shots. Find the distribution function of X.

Solution.

A discrete random variable X (the number of shots) has the following possible values:

x = 0 (the shooter never hit the target)
x = 1 (the shooter hit the target 1 time)
x = 2 (the shooter hit the target 2 times)
x = 3 (the shooter hit the target 3 times)

Let's find the probabilities of possible values of X using the Bernoulli's formula

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

By the condition of the task n=3; p=0,6; q=1-p=0,4. So, we get the result:

$$P_3(0) = q^3 = (0{,}4)^3 = 0{,}064$$
$$P_3(1) = C_3^1 \cdot p \quad \cdot q^2 = \frac{3!}{1! \cdot (3-1)!} \cdot 0{,}6 \cdot (0{,}4)^2 = 3 \cdot 0{,}6 \cdot 0{,}16 == 0{,}288$$
$$P_3(2) = C_3^2 \cdot p^2 \cdot q \quad = \frac{3!}{2! \cdot (3-2)!} \cdot (0{,}6)^2 \cdot 0{,}4 = 3 \cdot (0{,}6)^2 \cdot 0{,}4 = $$
$$= 0{,}432$$
$$P_3(3) = p^3 = (0{,}6)^3 = 0{,}216$$

Let's check the values of probabilities:

$$0{,}064 + 0{,}288 + 0{,}432 + 0{,}216 = 1$$

Let's write the sought binomial distribution law of the random variable X:

X	0	1	2	3
p	0,064	0,288	0,432	0,216

Let's find the distribution function.

If $x \leq 0 \; then \;\; F(x) = 0$. Really, $F(x) = P(X < 0) = 0$

{the random variable X does not take values less than zero}

If $0 < x \leq 1\ then\ \ F(x) = 0{,}064$. Really, $F(x) = P(X < 1) = 0{,}064$
{the random variable X can take the value x=0 with a probability p=0,064}

If $1 < x \leq 2\ then\ \ F(x) = 0{,}064 + 0{,}288 = 0{,}352$. Really, $F(x) = P(X < 2) = 0{,}064 + 0{,}288 = 0{,}352$
{the random variable X can take the value x=0 with a probability p=0,064 and it *can take the value x=1 with a probability p=0,288}*

If $2 < x \leq 3\ then\ \ F(x) = 0{,}064 + 0{,}288 + 0{,}432 = 0{,}784$. Really, $F(x) = P(X < 3) = 0{,}064 + 0{,}288 + 0{,}432 = 0{,}784$
{the random variable X can take the value x=0 with a probability p=0,064; it *can take the value x=1 with a probability p=0,288 and the probability that x=2 is equal to p=0,432}*

If $x > 3\ then\ \ F(x) = 1$. Really, *the random variable X can take the value x=0 with a probability p=0,064;* it *can take the value x=1 with a probability p=0,288; the probability that x=2 is equal to p=0,432 and the probability that x=3 is equal to p=0,216. We obtain that* $F(x) = 0{,}064 + 0{,}288 + 0{,}432 + 0{,}216 = 1$

So, the sought distribution function has the form:

$$F(x) = \begin{cases} 0, & x \leq 0 \\ 0{,}064, & 0 < x \leq 1 \\ 0{,}352 & 1 < x \leq 2 \\ 0{,}784 & 2 < x \leq 3 \\ 1, & x > 3 \end{cases}$$

Task № 4
A discrete random variable is given by the distribution law

X	2	4	7
p	0,5	0,2	0,3

Find the distribution function and plot it.
Solution.
If $x \leq 2\ then\ \ F(x) = 0$. Really, $F(x) = P(X < 2) = 0$
the random variable X does not take values less than two.

If $2 < x \leq 4\ then\ \ F(x) = 0{,}5$. Really, $F(x) = P(X < 4) = 0{,}5$
the random variable X can take the value x=2 with a probability p=0,5.

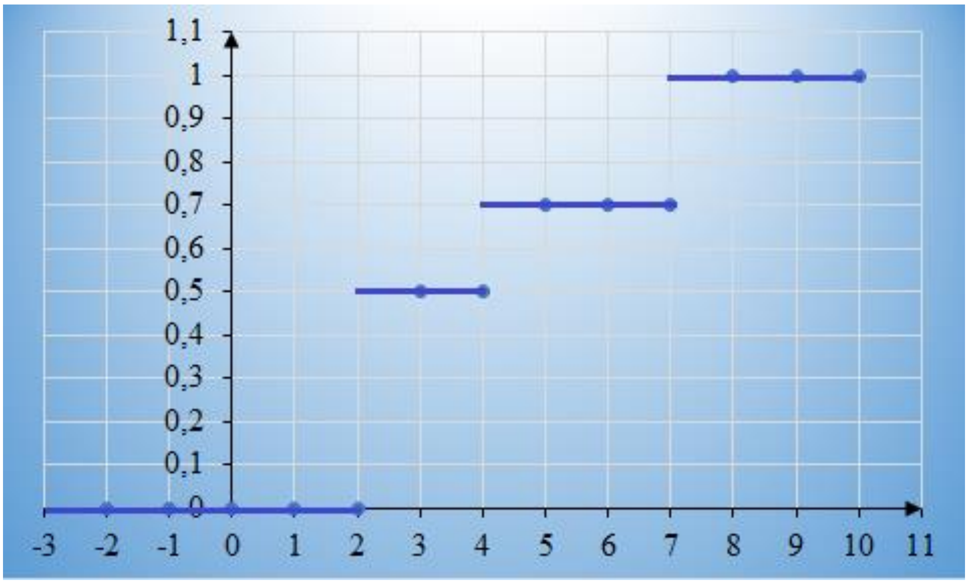
If $4 < x \leq 7$ *then* $F(x) = 0{,}5 + 0{,}2 = 0{,}7$. Really, $F(x) = P(X < 2) = 0{,}5 + 0{,}2 = 0{,}7$
the random variable X can take the value x=2 with a probability p=0,5 and it *can take the value x=4 with a probability p=0,2.*

If $x > 7$ *then* $F(x) = 1$. Really, *the random variable X can take the value x=2 with a probability p=0,5;* it *can take the value x=4 with a probability p=0,2; the probability that x=7 is equal to p=0,3. We obtain that* $F(x) = 0{,}5 + 0{,}2 + 0{,}3 = 1$

So, the sought distribution function has the form:

$$F(x) = \begin{cases} 0 & x \leq 2 \\ 0{,}5 & 2 < x \leq 4 \\ 0{,}7 & 4 < x \leq 7 \\ 1, & x > 7 \end{cases}$$

Let's plot its graph



Task № 5
A discrete random variable is given by the distribution law

X	3	4	7	10
p	0,2	0,1	0,4	0,3

Find the distribution function and plot it.
Solution.
If $x \leq 3$ *then* $F(x) = 0$. Really, $F(x) = P(X < 3) = 0$
the random variable X does not take values less than three.

If $3 < x \leq 4\ then\ \ F(x) = 0{,}2$. Really, $F(x) = P(X < 4) = 0{,}2$
the random variable X can take the value x=3 with a probability p=0,2.

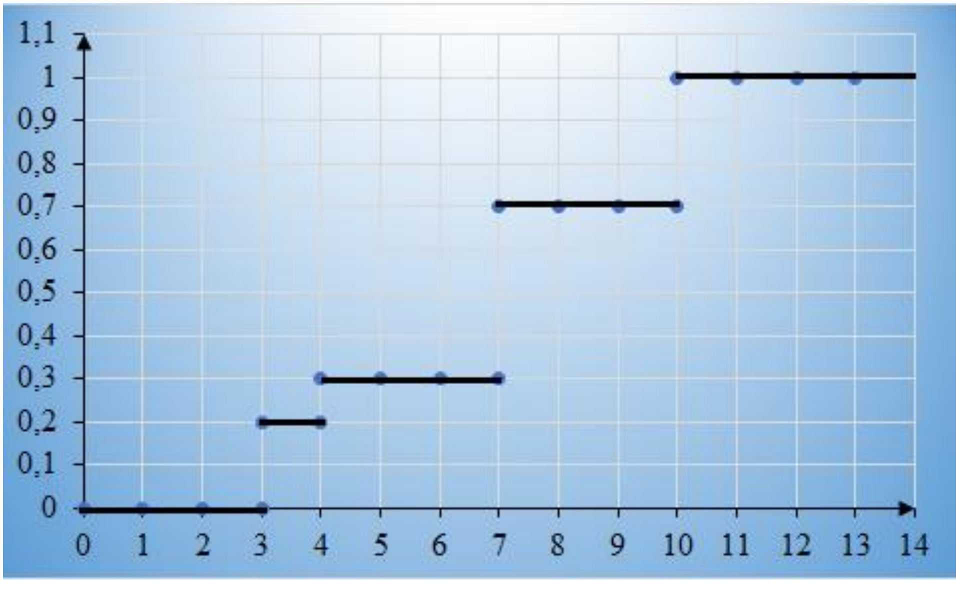
If $4 < x \leq 7\ then\ \ F(x) = 0{,}2 + 0{,}1 = 0{,}3$. Really, $F(x) = P(X < 7) = 0{,}2 + 0{,}1 = 0{,}3$
the random variable X can take the value x=3 with a probability p=0,2 and it *can take the value x=4 with a probability p=0,1.*

If $7 < x \leq 10\ then\ \ F(x) = 0{,}2 + 0{,}1 + 0{,}4 = 0{,}7$. Really, $F(x) = P(X < 10) = 0{,}2 + 0{,}1 + 0{,}4 = 0{,}7$
the random variable X can take the value x=3 with a probability p=0,2; it *can take the value x=4 with a probability p=0,1 and the probability that x=7 is equal to p=0,4.*

If $x > 10\ then\ \ F(x) = 1$. Really, *the random variable X can take the value x=3 with a probability p=0,2;* it *can take the value x=4 with a probability p=0,1; the probability that x=7 is equal to p=0,4 and the probability that x=10 is equal to p=0,3. We obtain that* $F(x) = 0{,}2 + 0{,}1 + 0{,}4 + 0{,}3 = 1$
So, the sought distribution function has the form:

$$F(x) = \begin{cases} 0 & x \leq 3 \\ 0{,}2 & 3 < x \leq 4 \\ 0{,}3 & 4 < x \leq 7 \\ 0{,}7 & 7 < x \leq 10 \\ 1, & x > 10 \end{cases}$$

Let's plot the graph of the distribution function



Task № 6
A random variable X is given by the distribution function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ x^2 & 0 < x \leq 1 \\ 1 & x > 10 \end{cases}$$

Find the probability that, as a result of four independent trials, the random variable X three times takes a value, that will belong to the interval (0,25; 0,75).

Solution.

$$P(0{,}25 < X < 0{,}75) = F(0{,}75) - F(0{,}25) = (0{,}75)^2 - (0{,}25)^2 = 0{,}5625 - 0{,}0625 = 0{,}5$$

Consider a random event A. It means that the value of the random variable X falls into the interval (0,25; 0,75).

We have independent trials, in each of which the probability of event A is constant and equal to p=0,5. We find the sought probability by the Bernoulli's formula

$$P_n(m) = C_n^m \cdot p^m \cdot q^{n-m}.$$

In accordance with the task, we can write n=4, m=3, p=0,5, q=1-p=0,5. The sought probability is:

$$P_4(3) = C_4^3 \cdot (0{,}5)^3 \cdot 0{,}5 = \frac{4!}{3!\,(4-3)!} \cdot (0{,}5)^4 = 4 \cdot 0{,}0625 = 0{,}25$$

Task № 7
A random variable X is given by a function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ Mx & 0 < x \leq 4 \\ 1 & x > 4 \end{cases}$$

Find 1) the value of the constant factor M; 2) the probability density; 3) the probability that X takes a value from the interval (1,2).

Solution.
To find the probability density function, let's use the formula

$$f(x) = F'(x)$$

We get the next result

$$f(x) = F'(x) = \begin{cases} 0 & x \leq 0 \\ M & 0 < x \leq 4 \\ 0 & x > 4 \end{cases}$$

To find the value of the constant factor M, let's use the formula

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

In our case this formula can be written in the form

$$\int_{0}^{4} f(x)dx = 1$$

We obtain:

$$\int_{0}^{4} f(x)dx = \int_{0}^{4} Mdx = Mx \Big|_0^4 = 4M$$

Let's compose and solve the equation

$$4M = 1 \Rightarrow M = \frac{1}{4}$$

Thus, we can write

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{1}{4} & 0 < x \leq 4 \\ 0 & x > 4 \end{cases}$$

Let's find the probability that X takes a value from the interval (1,2):

$$P(1 < X < 2) = F(2) - F(1) = \frac{1}{4} \cdot 2 - \frac{1}{4} \cdot 1 = \frac{1}{4}$$

Task № 8

A random variable is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \leq 0 \\ sinx & 0 < x \leq \pi/2 \\ 0 & x > \pi/2 \end{cases}$$

Find the integral distribution function F(x).

Solution.

Let's find the distribution function. We can use the formula

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

1. If $x \le 0$, then $f(x) = 0$

$$F(x) = \int_{-\infty}^{x} 0dx = 0$$

2. If $0 < x \le \pi/2$, then

$$F(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{x} sinxdx = -cosx \Big|_{0}^{x} = -cosx + cos0 = 1 - cosx$$

3. If $x > \pi/2$, then

$$F(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{\pi/2} sinxdx + \int_{\pi/2}^{\infty} 0dx = -cosx \Big|_{0}^{\frac{\pi}{2}}$$

$$= -\cos\left(\frac{\pi}{2}\right) + cos0 == 1$$

So, we get the result

$$F(x) = \begin{cases} 0 & x \le 0 \\ 1 - cosx & 0 < x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$$

Task № 9

A random variable is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 1 \\ Mx & 1 < x \le 3 \\ 0 & x > 3 \end{cases}$$

Find: 1) the constant factor M; 2) the integral distribution function; 3) the probability that X takes values from the interval (2,3); 4) plot the graph of the integral distribution function.

Solution.

1) To find the value of the constant factor M we use the formula

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

In our case this formula can be written in the form

$$\int_{1}^{3} f(x)dx = 1$$

We obtain:

$$\int_{1}^{3} f(x)dx = \int_{1}^{3} Mxdx = M\frac{x^2}{2}\Big|_1^3 = 4M$$

Let’s compose and solve the equation

$$4M = 1 \Rightarrow M = 1/4,$$

So, we can write

$$f(x) = \begin{cases} 0 & x \le 1 \\ x/4 & 1 < x \le 3 \\ 0 & x > 3 \end{cases}$$

2) Let’s find the distribution function. We can use the formula

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

1. If $x \le 1$, then $f(x) = 0$

$$F(x) = \int_{-\infty}^{x} 0dx = 0$$

2. If $1 < x \le 3$, then

$$F(x) = \int_{-\infty}^{1} 0dx + \int_{1}^{x} \frac{1}{4}xdx = \frac{1}{4}\int_{1}^{x} xdx = \frac{x^2}{8}\Big|_1^x = \frac{x^2}{8} - \frac{1}{8} = \frac{x^2 - 1}{8}$$

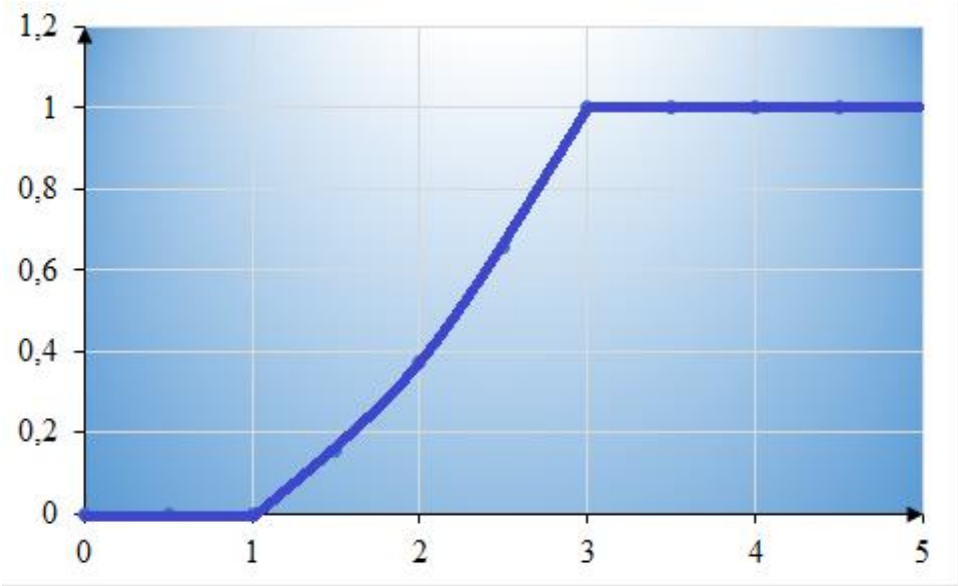
3. If $x > 3$, then

$$F(x) = \int_{-\infty}^{1} 0dx + \int_{1}^{3} \frac{1}{4}xdx + \int_{3}^{\infty} 0dx = \frac{x^2}{8}\Big|_1^3 = \frac{9}{8} - \frac{1}{8} = 1$$

$$F(x) = \begin{cases} 0 & x \le 1 \\ \dfrac{x^2 - 1}{8} & 1 < x \le 3 \\ 1 & x > 3 \end{cases}$$

3) $P(2 < X < 3) = F(3) - F(2) = \frac{x^2 - 1}{8}\Big|_{x=3} - \frac{x^2 - 1}{8}\Big|_{x=2} = 1 - \frac{3}{8} = \frac{5}{8}$

Let’s plot the graph of the integral distribution function.



Task № 10

A random variable is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 1 \\ M/3x & 1 < x \le e \\ 0 & x > e \end{cases}$$

Find: 1) the constant factor M; 2) the integral distribution function F(x) and plot
its graph.

Solution.

1) To find the value of the constant factor M, we use the formula

$$\int_{-\infty}^{+\infty} f(x)dx = 1$$

In our case this formula can be written in the form

$$\int_{1}^{e} f(x)dx = 1$$

We obtain:

$$\int_{1}^{e} f(x)dx = \int_{1}^{e} \frac{M}{3x}dx = \frac{M}{3}\int_{1}^{e}\frac{dx}{x} = \frac{M}{3}\ln|x|\Big|_{1}^{e} = \frac{M}{3}(lne - ln1) = \frac{M}{3}$$

Let’s compose and solve the equation

$$\frac{M}{3} = 1 \Rightarrow M = 3$$

2) Let’s find the distribution function. We can use the formula

$$F(x) = \int_{-\infty}^{x} f(x)dx$$

1. If $x \le 1$, then $f(x) = 0$

$$F(x) = \int_{-\infty}^{x} 0dx = 0$$

2. If $1 < x \le e$, then

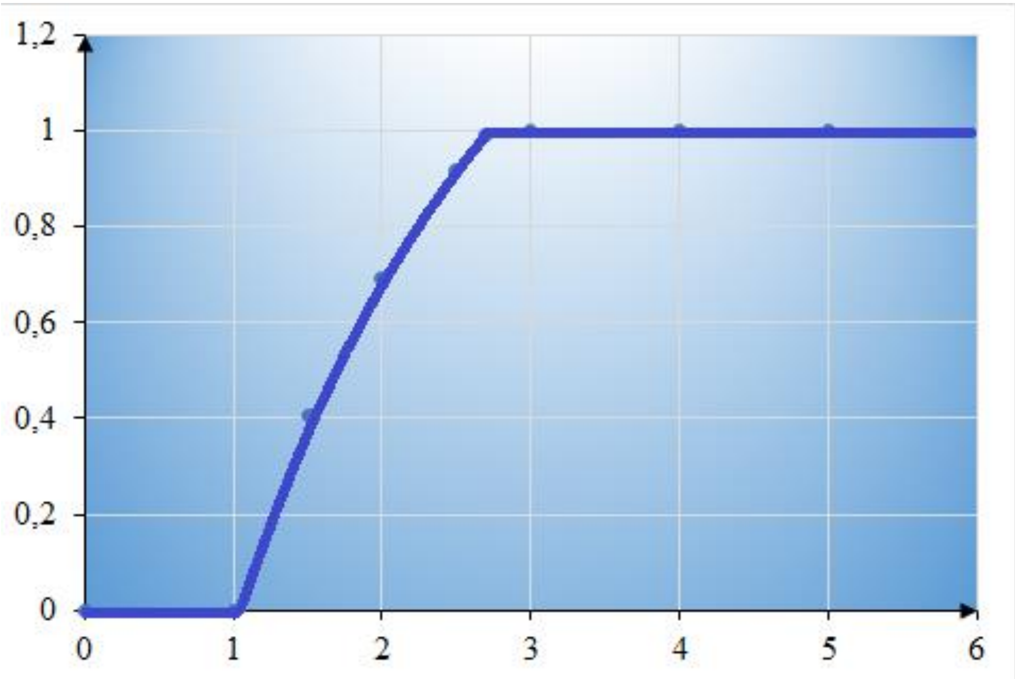
$$F(x) = \int_{-\infty}^{1} 0dx + \int_{1}^{x} \frac{3}{3x}dx = \int_{1}^{x} \frac{dx}{x} = \ln|x|\Big|_1^x = \ln|x| - \ln|1| = \ln|x|$$

3. If $x > e$, then

$$F(x) = \int_{-\infty}^{1} 0dx + \int_{1}^{e} \frac{dx}{x} + \int_{e}^{\infty} 0dx = \ln|x|\Big|_1^e = \ln e - \ln 1 = 1$$

$$F(x) = \begin{cases} 0 & x \le 1 \\ lnx & 1 < x \le e \\ 1 & x > e \end{cases}$$

Let’s plot the graph of the integral distribution function



PRACTICAL TASKS FOR INDEPENDENT WORK

1. A random variable is given by the distribution density function

$$f(x)=\begin{cases} 0 & x\le 1 \\ x-1/2 & 1<x\le 2 \\ 0 & x>2 \end{cases}$$

Find the integral distribution function F(x).

Answer:

$$F(x)=\begin{cases} 0 & x\le 1 \\ \dfrac{x^2-x}{2} & 0<x\le 2 \\ 1 & x>2 \end{cases}$$

2. A random variable is given by the distribution density function

$$f(x)=\begin{cases} 0 & x\le \pi/6 \\ 3sin3x & \pi/6<x\le \pi/3 \\ 0 & x>\pi/3 \end{cases}$$

Find the integral distribution function F(x).

Answer:

$$F(x)=\begin{cases} 0 & x\le \pi/6 \\ -cos3x & \pi/6<x\le \pi/3 \\ 1 & x>\pi/3 \end{cases}$$

3. A random variable X is given by the integral distribution function

$$F(x)=\begin{cases} 0 & x\le 0 \\ tg\dfrac{x}{8} & 0<x\le 2\pi \\ 1 & x>2\pi \end{cases}$$

Find: 1) the differential function; 2) the probability that X takes values from the interval $(\pi, 3\pi)$.

Answer:

1) $$f(x)=F'(x)=\begin{cases} 0 & x\le 0 \\ \frac{1}{8cos^2\frac{x}{8}} & 0<x\le 2\pi \\ 1 & x>2\pi \end{cases}$$

2) $$P(\pi<X<3\pi)=F(3\pi)-F(\pi)=1-\ tg\frac{\pi}{8}$$

4. A random variable X is given by the integral distribution function

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{Cos2x - 1}{M} & 0 < x \le \pi/2 \\ 1 & x > \pi/2 \end{cases}$$

Find the value of the constant factor M.

Answer: $M = -2$

SELF-TEST QUESTIONS

1. The integral distribution function is given:

$$F(x) = \begin{cases} 0, & x \le 1 \\ \dfrac{x-1}{2}, & 1 < x \le 3 \\ 1, & x > 3 \end{cases}$$

What is the value of the differential function at the point $x = 2$?

   a. $1/3$
   b. $1/2$
   c. $1/4$

2. The distribution density is given:

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2 + \dfrac{2}{3}, & 0 < x \le 1 \\ 0, & x > 1 \end{cases}$$

What is the value of the integral function at the point $x = 1/2$?

   a. $1/20$
   b. $1/22$
   c. $9/24$

3. A random variable is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 4 \\ Mx & 4 < x \le 5 \\ 0 & x > 5 \end{cases}$$

What is the constant factor M?

   a. $2/9$
   b. $9/5$

c. $5/9$

4. A continuous random variable X is given by a distribution function:

$$F(x)=\begin{cases}0, & x\le 0\\ sin2x, & 0<x\le \pi/4\\ 1, & x>\pi/4\end{cases}$$

What is the probability that X takes a value in the interval $(\pi/12, \pi/8)$?

a. $(\sqrt{2}+1)/2$
b. $(\sqrt{2}-1)/2$
c. $(1-\sqrt{2})/2$

5. A discrete random variable is given by the distribution law

X	1	2	3
p	0,1	0,3	0,6

What is the value of the distribution function at the point $x = 2{,}5$?

a. 0,3
b. 0,4
c. 0,5

LITERATURE FOR SELF-STUDY

1. V. E. Gmurman, I. I Berenblut Fundamentals of probability theory and mathematical statistics. [Electronic edition]
Access mode: https://www.worldcat.org/title/fundamentals-of-probability-theory-and-mathematical-statistics/oclc/499092704?referer=di&ht=edition
2. Joseph. K. Blitzstein, J. Hwang Introduction to Probability. Second edition. Chapman and Hall/CRC. 2019. 634 p.
3. S.P. Thompson A textbook of higher mathematics: learning calculus in a simple way. Access mode: https://www.amazon.com/TextBook-Higher-Mathematics-Integration-Differentiation-ebook/dp/B074TVTQ32

TOPIC 6. NUMERICAL CHARACTERISTICS OF RANDOM VARIABLES

PLAN

1. Numerical characteristics of the discrete random variable.
2. Numerical characteristics of the continuous random variable.

Key notions, definitions and categories to be studied: *The concept of the mathematical expectation, dispersion and standard deviation of the discrete and continuous random variables.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

A random variable X is given by the distribution law

X	-2	-1	0	3
p	0,1	0,5	0,1	0,3

Find $M(x),\ D(x),\ \sigma(x)$.

Solution.

The mathematical expectation $M(x)$ we can calculate by the formula

$$M(x) = x_1p_1 + x_2p_1 + \cdots + x_np_n$$

We obtain the value of the mathematical expectation

$$M(x) = -2 \cdot 0{,}1 - 1 \cdot 0{,}5 + 0 \cdot 0{,}1 + 3 \cdot 0{,}3 = 0{,}2$$

The dispersion $D(x)$ we can calculate by the formula

$$D(X) = M(X^2) - [M(X)]^2$$

Let's find $M(X^2)$ and $[M(X)]^2$:

$M(X^2) = (-2)^2 \cdot 0{,}1 + (-1)^2 \cdot 0{,}5 + 3^2 \cdot 0{,}3 = 3{,}6$

$[M(X)]^2 = (0{,}2)^2 = 0{,}04$

So, we get the value of the dispersion

$$D(X) = M(X^2) - [M(X)]^2 = 3{,}6 - 0{,}04 = 3{,}56$$

The standard deviation we can calculate by the formula

$$\sigma(x) = \sqrt{D(X)}$$
$$\sigma(x) = \sqrt{3{,}56} \approx 1{,}89$$

Task № 2

Find $M(3x - y)$, if $M(x) = 2$, $M(y) = 1$.

Solution.

Let's use the properties of the mathematical expectation:

1. The mathematical expectation of the sum of random variables is equal to the sum of the mathematical expectations of the terms:

$$\mathrm{M}(\mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n) = \mathrm{M}(\mathrm{X}_1) + \mathrm{M}(\mathrm{X}_2) + \cdots + \mathrm{M}(\mathrm{X}_n)$$

2. A constant multiplier can be taken out of a sign of the mathematical expectation:

$$\mathrm{M}(\mathrm{CX}) = \mathrm{CM}(\mathrm{X}), \qquad \mathrm{C} - const$$

So, we get the next result

$$M(3x - y) = M(3x) - M(y) = 3M(x) - M(y) = 3 \cdot 2 - 1 = 5$$

Task № 3

Find $D(2x + 3y)$, if $D(x) = 4$, $D(y) = 2$.

Solution.

Let's use the properties of the dispersion:

1. The dispersion of the sum of independent random variables is equal to the sum of the dispersions of the terms:

$$D(\mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n) = D(\mathrm{X}_1) + D(\mathrm{X}_2) + \cdots + D(\mathrm{X}_n)$$

2. A constant multiplier can be taken out of the dispersion sign. There is a formula:

$$D(\mathrm{CX}) = C^2 D(\mathrm{X}), \qquad \mathrm{C} - const$$

So, we get the result

$$D(2x + 3y) = D(2x) + D(3y) = 4D(x) + 9D(y) = 4 \cdot 4 + 9 \cdot 2 = 34$$

Task № 4

A random variable X is given by the distribution density function

$$f(x)=\begin{cases}0 & x\le 0\\ \frac{3}{2}\sqrt{x} & 0<x\le 1\\ 0 & x>1\end{cases}$$

Find $M(x),\ D(x),\ \sigma(x)$.

Solution.

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X)=\int_a^b xf(x)\,dx$$

By the condition of the task a=0, b=1, $f(x)=\frac{3}{2}\sqrt{x}$. We obtain the value of the mathematical expectation

$$M(X)=\int_0^1 x\frac{3}{2}\sqrt{x}\,dx=\frac{3}{2}\int_0^1 x^{3/2}dx=\frac{3}{5}x^{5/2}\Big|_0^1=\frac{3}{5}\sqrt{x^5}\Big|_0^1=\frac{3}{5}=0{,}6$$

The dispersion of a continuous random variable X can be calculated by the formula

$$D(X)=\int_a^b x^2f(x)\,dx-\ [M(X)]^2$$

So, we obtain the value of the dispersion

$$D(X)=\int_0^1 x^2\cdot\frac{3}{2}\sqrt{x}dx-\left(\frac{3}{5}\right)^2=\frac{3}{2}\int_0^1 x^{\frac{5}{2}}\,dx-\frac{9}{25}=$$

$$=\frac{3}{7}x^{\frac{7}{2}}\Big|_0^1-\frac{9}{25}=\frac{3}{7}-\frac{9}{25}=\frac{75-63}{175}=\frac{12}{175}$$

The standard deviation we can calculate by the formula

$$\sigma(x)=\sqrt{D(X)}$$

$$\sigma(x)=\sqrt{\frac{12}{175}}\approx 0{,}26$$

Task № 5

A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{2}x & 0 < x \le 2 \\ 0 & x > 2 \end{cases}$$

Find $M(x),\ D(x),\ \sigma(x)$.

Solution.

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X) = \int_a^b xf(x)\,dx$$

By the condition of the task we can write: a=0, b=2, $f(x) = \frac{1}{2}x$. We obtain the next value of the mathematical expectation

$$M(X) = \int_0^2 x\frac{1}{2}x\,dx = \frac{1}{2}\int_0^2 x^2dx = \frac{1}{6}x^3\,\Big|_0^2 = \frac{4}{3}$$

The dispersion of a continuous random variable X can be calculated by the formula

$$D(X) = \int_a^b x^2 f(x)\,dx - [M(X)]^2$$

So, we obtain the value of the dispersion

$$D(X) = \int_0^2 x^2 \cdot \frac{1}{2}xdx - \left(\frac{4}{3}\right)^2 = \frac{1}{2}\int_0^2 x^3\,dx - \frac{16}{9} =$$

$$= \frac{1}{8}x^4\,\Big|_0^2 - \frac{16}{9} = 2 - \frac{16}{9} = \frac{2}{9}$$

The standard deviation we can calculate by the formula

$$\sigma(x) = \sqrt{D(X)}$$

$$\sigma(x) = \sqrt{\frac{2}{9}} \approx 0{,}47$$

Task № 6

A random variable X is given by the distribution density function

$$f(x)=\begin{cases}0 & x\le 0\\ \frac{1}{4}sinx & 0<x\le\pi\\ 0 & x>\pi\end{cases}$$

Find the mathematical expectation $M(x)$.

Solution.

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X)=\int_a^b xf(x)\,dx$$

By the condition of the task we can write: a=0, b=$\pi, f(x)=\frac{1}{4}sinx$. We obtain the next value of the mathematical expectation

$$M(X)=\int_0^\pi x(\frac{1}{4}sinx)\,dx=\frac{1}{4}\int_0^\pi xsinx\,dx=\left\{\begin{matrix}u=x & dv=sinxdx\\ du=dx & v=-cosx\end{matrix}\right\}$$

$$=\frac{1}{4}\left(-xcosx\Big|_0^\pi+\int_0^\pi cosxdx\right)=\frac{1}{4}\left(\pi+sinx\Big|_0^\pi\right)=\frac{\pi}{4}$$

Task № 7

A random variable X is given by the distribution density function

$$f(x)=\begin{cases}0 & x\le 3\\ -\frac{3}{4}x^2+6x-\frac{45}{4} & 3<x\le 5\\ 0 & x>5\end{cases}$$

Find the mathematical expectation $M(x)$.

Solution.

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X)=\int_a^b xf(x)\,dx$$

By the condition of the task we can write: a=3, b= 5, $f(x) = -\frac{3}{4}x^2 + 6x - \frac{45}{4}$. We obtain the next value of the mathematical expectation

$$M(X) = \int_3^5 x(-\frac{3}{4}x^2 + 6x - \frac{45}{4})\,dx = \int_3^5 (-\frac{3}{4}x^3 + 6x^2 - \frac{45}{4}x)\,dx =$$

$$= \left(-\frac{3}{16}x^4 + 2x^3 - \frac{45}{8}x^2\right)\Big|_3^5 = -\frac{3}{16}\cdot 5^4 + 2\cdot 5^3 - \frac{45}{8}\cdot 5^2$$

$$+\frac{3}{16}\cdot 3^4 - 2\cdot 3^3 + \frac{45}{8}\cdot 3^2 =$$

$$= -\frac{1875}{16} + 250 - \frac{1125}{8} + \frac{243}{16} - 54 + \frac{405}{8} =$$

$$= -\frac{1632}{16} - \frac{720}{8} + 196 = -\frac{3072}{16} + 196 = -192 + 196 = 4$$

Task № 8

A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le -2 \\ \frac{1}{4}x + \frac{1}{2} & -2 < x \le 2 \\ 0 & x > 2 \end{cases}$$

Find $M(x)$, $D(x)$, $\sigma(x)$

Solution.

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X) = \int_a^b xf(x)\,dx$$

By the condition of the task we can write: a=-2, b= 2, $f(x) = \frac{1}{4}x + \frac{1}{2}$

$$M(X) = \int_{-2}^{2} x(\frac{1}{4}x + \frac{1}{2})\,dx =$$

$$= \frac{1}{4}\int_{-2}^{2} x^2 dx + \frac{1}{2}\int_{-2}^{2} x dx = \frac{1}{12}x^3\Big|_{-2}^{2} + \frac{1}{4}x^2\Big|_{-2}^{2} = \frac{8}{12} + \frac{8}{12}$$

$$+ 1 \quad - 1 = \frac{4}{3}$$

The dispersion of a continuous random variable X can be calculated by the formula

$$D(X) = \int_{a}^{b} x^2 f(x)\,dx - [M(X)]^2$$

So, we obtain the value of the dispersion

$$D(X) = \int_{-2}^{2} x^2 \cdot \left(\frac{1}{4}x + \frac{1}{2}\right) dx - \left(\frac{4}{3}\right)^2 = \frac{1}{4}\int_{-2}^{2} x^3\,dx + \frac{1}{2}\int_{-2}^{2} x^2\,dx - \frac{16}{9} =$$

$$= \frac{1}{16}x^4 \Big|_{-2}^{2} + \frac{1}{6}x^3 \Big|_{-2}^{2} - \frac{16}{9} = \frac{1}{16}(16-16) + \frac{1}{6}(8+8) - \frac{16}{9}$$

$$= \frac{8}{3} - \frac{16}{9} = \frac{8}{9}$$

The standard deviation we can calculate by the formula

$$\sigma(x) = \sqrt{D(X)}$$

$$\sigma(x) = \sqrt{\frac{8}{9}} \approx 0{,}94$$

Task № 9

A random variable X is given by the distribution function

$$F(x) = \begin{cases} 0 & x \le 2 \\ \dfrac{x-2}{3} & 2 < x \le 5 \\ 1 & x > 5 \end{cases}$$

Find $M(x),\ D(x),\ \sigma(x)$

Solution.

Let’s find the distribution density

$$f(x) = F'(x) = \begin{cases} 0 & x \le 2 \\ 1/3 & 2 < x \le 5 \\ 0 & x > 5 \end{cases}$$

The mathematical expectation $M(x)$ of a continuous random variable X we can calculate by the formula

$$M(X)=\int_a^b xf(x)\,dx$$

By the condition of the task we can write: a=2, b= 5, $f(x)=1/3$

$$M(X)=\frac{1}{3}\int_2^5 x\,dx=\frac{x^2}{6}\bigg|_2^5=\frac{25}{6}-\frac{4}{6}=\frac{21}{6}=\frac{7}{2}$$

The dispersion of a continuous random variable X can be calculated by the formula

$$D(X)=\int_a^b x^2f(x)\,dx-\ [M(X)]^2$$

So, we obtain the value of the dispersion

$$D(X)=\frac{1}{3}\int_2^5 x^2\,dx-\left(\frac{7}{2}\right)^2=\frac{x^3}{9}\bigg|_2^5-\frac{49}{4}=\frac{125}{9}-\frac{8}{9}-\frac{49}{4}=\frac{117}{9}-\frac{49}{4}=$$
$$=\frac{468-441}{36}=\frac{27}{36}=\frac{3}{4}$$

The standard deviation we can calculate by the formula

$$\sigma(x)=\sqrt{D(X)}$$
$$\sigma(x)=\sqrt{\frac{3}{4}}\approx 0{,}87$$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. A random variable X is given by the distribution law

X	-2	-1	0	3
p	0,1	0,5	0,1	0,3

Find $M(x),\ D(x),\ \sigma(x)$.

Answer: $M(x)=0{,}2$; $D(X)=3{,}56$; $\sigma(x)\approx 1{,}89$

2. A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{3}{2}\sqrt{x} & 0 < x \le 1 \\ 0 & x > 1 \end{cases}$$

Find $M(x)$, $\sigma(x)$.

Answer: $M(x) = 0{,}6$; $\sigma(x) \approx 0{,}26$

3. A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{2}x & 0 < x \le 2 \\ 0 & x > 2 \end{cases}$$

Find $M(x)$, $D(x)$.

Answer: $M(x) = 4/3$; $D(x) = 2/9$

4. A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{1}{4}sinx & 0 < x \le \pi \\ 0 & x > \pi \end{cases}$$

Find $M(x)$

Answer: $M(x) = \pi/4$

5. A random variable X is given by the distribution function

$$F(x) = \begin{cases} 0 & x \le 2 \\ \dfrac{x-2}{3} & 2 < x \le 5 \\ 0 & x > 5 \end{cases}$$

Find $M(x), D(x)$.

Answer: $M(x) = 3{,}5$; $D(x) = 0{,}75$

SELF-TEST QUESTIONS

1. A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \le 0 \\ Cosx & 0 < x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}$$

What is the mathematical expectation of the random variable X?

a. $(\pi - 2)/2$

b. $(\pi - 1)/2$
c. $(1 - \pi)/2$

2. A random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0 & x \leq -2 \\ \dfrac{1}{4}x + \dfrac{1}{4}, & -2 < x \leq 2 \\ 0 & x > 2 \end{cases}$$

What is the mathematical expectation of the random variable X?

a. $3/4$
b. $4/3$
c. $2/3$

3. A random variable X can take on the following values: 1, 2, 3 with probabilities of 0,1; 0,2; 0,3 respectively. What is the mathematical expectation of the random variable X?

a. 1,4
b. 2
c. 2,5

4. The standard deviation of the random variable X is equal to $\sigma(X) = 15$. Find its dispersion D(X)

a. 169
b. 225
c. 100

5. Find the standard deviation of the random variable X, if its dispersion D(X) is equal to 4

a. 2
b. 1
c. 0
d. another value

LITERATURE FOR SELF-STUDY

1. V. E. Gmurman, I. I Berenblut Fundamentals of probability theory and mathematical statistics. [Electronic edition]
Access mode: https://www.worldcat.org/title/fundamentals-of-probability-theory-and-mathematical-statistics/oclc/499092704?referer=di&ht=edition
2. Prasanna Sahoo Probability and Mathematical Statistics. University of Louisville. 2015. 698 p.
3. S.P. Thompson A textbook of higher mathematics: learning calculus in a simple way. Access mode: https://www.amazon.com/TextBook-Higher-Mathematics-Integration-Differentiation-ebook/dp/B074TVTQ32

TOPIC 7. DISTRIBUTION LAWS OF RANDOM VARIABLES

PLAN

1. Distribution laws of discrete random variables.
2. Distribution laws of continuous random variables.
3. Plotting integral and differential functions of normal, exponential, uniform distribution.

Key notions, definitions and categories to be studied: *The concept of the laws of distribution of random variables and the numerical characteristics of these variables.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1
The probability of winning one lottery ticket is 0.2. Bought 20 tickets. Find:
1) the mathematical expectation (average number) of the number of winning tickets;
2) the dispersion of the number of winning tickets.
Solution.
Let random variable X is the number of wins. Ticket wins are independent events, so random variable X is distributed according to the binomial law. *The mathematical expectation of a random variable X* distributed by the binomial law is equal to

$$M(X) = np$$

The dispersion of a random variable X distributed by the binomial law is equal to

$$D(X) = npq$$

By the condition of the task we can write: n=20; p=0,2; q=1-p=1-0,2=0,8.
Obtain the values of mathematical expectation and dispersion of X:

$$M(X) = np = 20 \cdot 0{,}2 = 4$$

$$D(X) = npq = 20 \cdot 0{,}2 \cdot 0{,}8 = 3{,}2$$

Task № 2

The probability of one non-standard detail is equal to 0,003. Find: 1) the average number of defective details from a batch of 3000 details; 2) the dispersion of the number of defective details.

Solution.
Let random variable X is the number of defective details. The probability of finding a non-standard detail is small enough and the number of details is large enough. So, random variable X is distributed according to the Poisson's law. *The mathematical expectation of a random variable X* distributed by the Poisson's law is equal to

$$M(X) = np$$

The dispersion of a random variable X distributed by the Poisson's law is equal to

$$D(X) = np$$

By the condition of the task we can write: n=3000; p=0,003; q=1-p=1-0,003 = 0,997. Obtain the values of mathematical expectation and dispersion of X:

$$M(X) = np = 3000 \cdot 0{,}003 = 9$$
$$D(X) = np = 3000 \cdot 0{,}003 = 9$$

Task № 3
The store received 1000 bottles of mineral water. The probability that the bottle will be broken during transportation is 0.002. Find: 1) the average number (mathematical expectation) of the number of bottles broken during transportation; 2) the dispersion of the number of broken bottles.

Solution.
Let random variable X is the number of broken bottles. The probability that the bottle will be broken during transportation is small enough and the number of bottles is large enough. So, random variable X is distributed according to the Poisson's law. *The mathematical expectation of a random variable X* distributed by the Poisson's law is equal to

$$M(X) = np$$

The dispersion of a random variable X distributed by the Poisson's law is equal to

$$D(X) = np$$

By the condition of the task we can write: n=1000; p=0,002; q=1-p=1-0,002=0,998.
Obtain the values of mathematical expectation and dispersion of X:

$$M(X) = np = 1000 \cdot 0{,}002 = 2$$
$$D(X) = np = 1000 \cdot 0{,}002 = 2$$

Task № 4

Continuous random variable X is given by the distribution function

$$F(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{1}{9}x & 0 < x \leq 9 \\ 1 & x > 9 \end{cases}$$

Find: 1) the distribution density function f(x); 2) mathematical expectation M(x), dispersion D(x), standard deviation $\sigma(X)$; 3) plot graphs of functions F (x) and f(x).

Solution.

Let's find distribution density function:

$$f(x) = F'(x) = \begin{cases} 0 & x \leq 0 \\ \dfrac{1}{9} & 0 < x \leq 9 \\ 0 & x > 9 \end{cases}$$

Let's compare the form of the obtained distribution density function with the uniform distribution density function

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & if \;\; x \epsilon (a, b) \\ 0, & if \;\; x \notin (a, b) \end{cases}$$

In our case

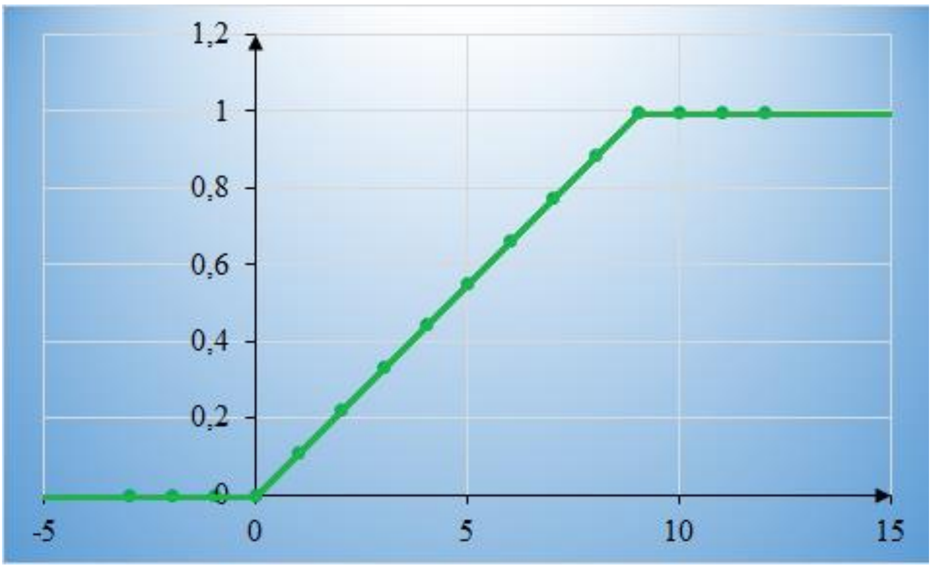
$$f(x) = \begin{cases} \dfrac{1}{9}, & if \;\; x \epsilon (0, 9) \\ 0, & if \;\; x \notin (0, 9) \end{cases}$$

Thus, we can conclude, that the random variable X has the uniform distribution and we can find its numerical characteristics using the formulas

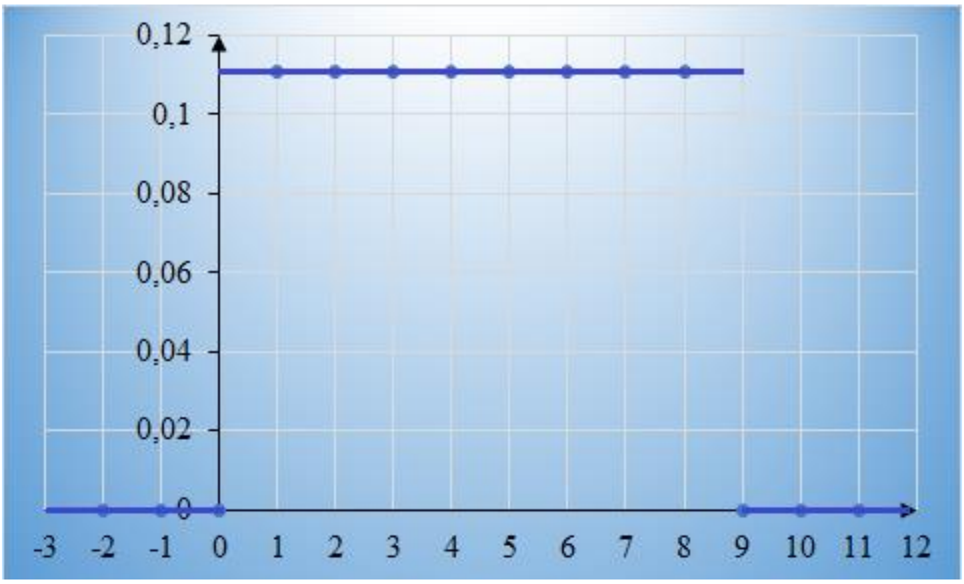
$$M(X) = \frac{a+b}{2} \qquad D(X) = \frac{(b-a)^2}{12}, \qquad \sigma(X) = \sqrt{D(X)} = \frac{b-a}{2\sqrt{3}}.$$

$$M(X) = \frac{0+9}{2} = 4{,}5 \qquad D(X) = \frac{(0-9)^2}{12} = 6{,}75 \qquad \sigma(X) = \sqrt{D(X)} \approx 2{,}60$$

Let's plot graphs of functions $F(x)$, $f(x)$. This is the graph of the function $F(x)$



And this is the graph of the function $f(x)$



Task № 5

A continuous random variable X is given by the distribution function

$$F(x) = \begin{cases} 0 & x \le 3 \\ Ax - 3 & 3 < x \le 4 \\ 1 & x > 4 \end{cases}$$

Find: 1) the constant value A; 2) the distribution function f(x); 3) the mathematical expectation M(x), dispersion D(x) and standard deviation $\sigma(x)$.

Solution.

Let's find the distribution density:

$$f(x) = F'(x) = \begin{cases} 0 & x \le 3 \\ A & 3 < x \le 4 \\ 0 & x > 4 \end{cases}$$

Let’s compare the form of the obtained distribution density function with the uniform distribution density function

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & if \ \ x \epsilon (a, b) \\ 0, & if \ \ x \notin (a, b) \end{cases}$$

In our case

$$f(x) = \begin{cases} A, & if \ \ x \epsilon (3, 4) \\ 0, & if \ \ x \notin (3, 4) \end{cases}$$

Let’s make the equation:

$$\frac{1}{b-a} = \mathrm{A} \Rightarrow \mathrm{A} = \frac{1}{4-3} = 1$$

The distribution density function has the form

$$f(x) = \begin{cases} 0 & x \le 3 \\ 1 & 3 < x \le 4 \\ 0 & x > 4 \end{cases}$$

Thus, in view of the fact that the random variable X has the uniform distribution, we can find its numerical characteristics:

$$M(X) = \frac{a+b}{2} \qquad D(X) = \frac{(b-a)^2}{12}, \qquad \sigma(X) = \sqrt{D(X)} = \frac{b-a}{2\sqrt{3}}$$

$$M(X) = \frac{3+4}{2} = \frac{7}{2} \qquad D(X) = \frac{(4-3)^2}{12} = \frac{1}{12} \qquad \sigma(X) = \frac{4-3}{2\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

Task № 6

A continuous random variable X has the uniform distribution law on the interval (4, 9). Find: 1) $M(X), D(X), \sigma(X)$; 2) write the distribution density function;

3) write the distribution function; 4) find the probability that a random variable X belongs to the interval (6, 10).

Solution.

1) A continuous random variable X has a uniform distribution law on the interval (4, 9), so we can calculate numerical characteristics of X by the formulas:

$$M(X) = \frac{a+b}{2} \qquad D(X) = \frac{(b-a)^2}{12}, \qquad \sigma(X) = \sqrt{D(X)} = \frac{b-a}{2\sqrt{3}}$$

We get the next result:

$$M(X) = \frac{4+9}{2} = \frac{13}{2} \qquad D(X) = \frac{(9-4)^2}{12} = \frac{25}{12} \qquad \sigma(X) = \frac{9-4}{2\sqrt{3}} = \frac{5}{2\sqrt{3}}$$

2) Write the distribution density function

$$f(x) = \begin{cases} 0 & x \le 4 \\ \frac{1}{5} & 4 < x \le 9 \\ 0 & x > 9 \end{cases}$$

3) Write the distribution function

When $x \le 4$

$$F(x) = \int_{-\infty}^{x} 0dx = 0$$

When $4 < x \le 9$

$$F(x) = \int_{-\infty}^{4} 0dx + \int_{4}^{x} \frac{1}{5}dx = \frac{1}{5}\int_{4}^{x} dx = \frac{x}{5}\Big|_4^x = \frac{x}{5} - \frac{4}{5} = \frac{x-4}{5}$$

When $x > 9$

$$F(x) = \int_{-\infty}^{4} 0dx + \int_{4}^{9} \frac{1}{5}dx + \int_{9}^{\infty} 0dx = \frac{1}{5}\int_{4}^{9} dx = \frac{1}{5}x\Big|_4^9 = \frac{9}{5} - \frac{4}{5} = 1$$

So, we can write

$$F(x) = \begin{cases} 0 & x \le 4 \\ \frac{x-4}{5} & 4 < x \le 9 \\ 1 & x > 9 \end{cases}$$

4) The probability that a random variable distributed according to a uniform law will belong to the interval (a, b) is equal to

$$P(a < X < b) = F(b) - F(a)$$

In our case a=6, b=10. So, using the formula we can write

$$P(6 < X < 10) = F(10) - F(6) = 1 - \frac{x-4}{5}\Big|_{x=6} = 1 - \frac{6-4}{5} = 1 - \frac{2}{5} = \frac{3}{5}$$

Task № 7

A continuous random variable X is given by a distribution function

$$F(x) = \begin{cases} 0, & if \quad x < 0 \\ 1 - e^{-3x}, & if \ \ x \geq 0 \end{cases}$$

Find: 1) the distribution density; 2) the mathematical expectation M(x), dispersion D(x), standard deviation $\sigma(x)$; 3) plot graphs of functions F (x), f(x).

Solution.

1) The distribution density is

$$f(x) = F'(x) = \begin{cases} 0, & if \quad x < 0 \\ 3e^{-3x}, & if \ \ x \geq 0 \end{cases}$$

2) Let’s compare the form of the given distribution function with the exponential distribution function

$$F(x) = \begin{cases} 0, & if \quad x < 0 \\ 1 - e^{-\lambda x}, & if \ \ x \geq 0 \end{cases}$$

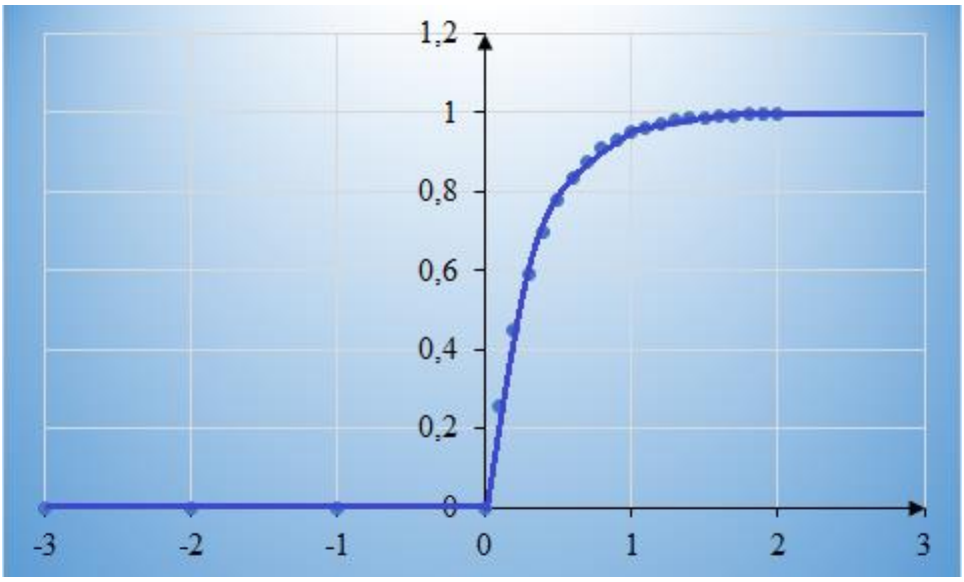
We can conclude, that the random variable X has an exponential distribution law. Looking at the form of writing the distribution function, we can conclude that λ = 3.

3) Thus, numerical characteristics of a random variable X we can calculate by the formulas:

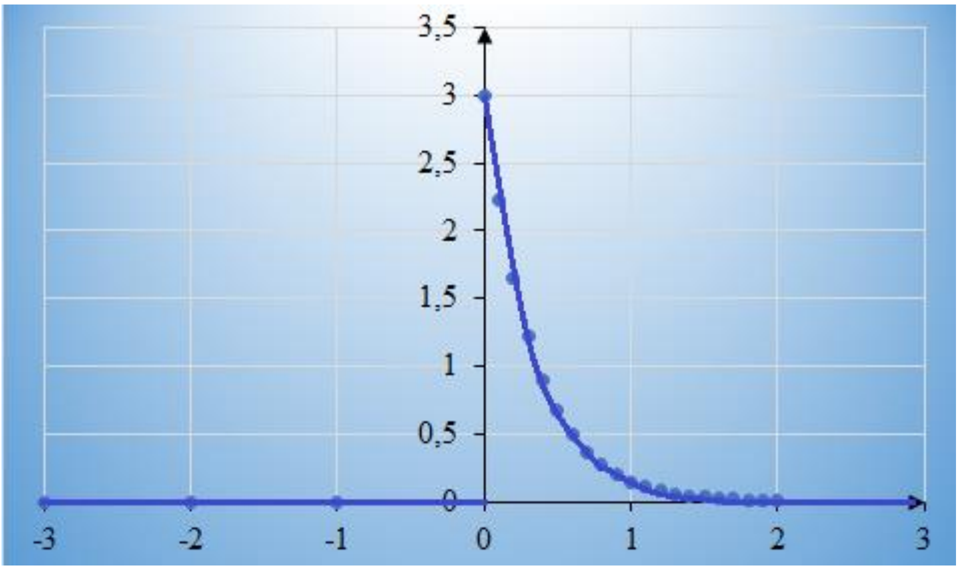
$$M(X) = \frac{1}{\lambda} \qquad D(X) = \frac{1}{\lambda^2}, \qquad \sigma(X) = \sqrt{D(X)} = \frac{1}{\lambda}$$

$$M(X) = \frac{1}{3} \qquad D(X) = \frac{1}{3^2} = \frac{1}{9} \qquad \sigma(X) = \sqrt{D(X)} = \frac{1}{3}$$

4) Let’s plot graphs of functions $F(x)$, $f(x)$. This is the graph of the function $F(x)$



And this is the graph of the function $f(x)$



Task № 8

The mathematical expectation of the random variable X is equal to 5, and the dispersion of X is equal to 9. Write down the probability density. Plot its graph.

Solution.

The distribution density of a continuous random variable X has the form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$$

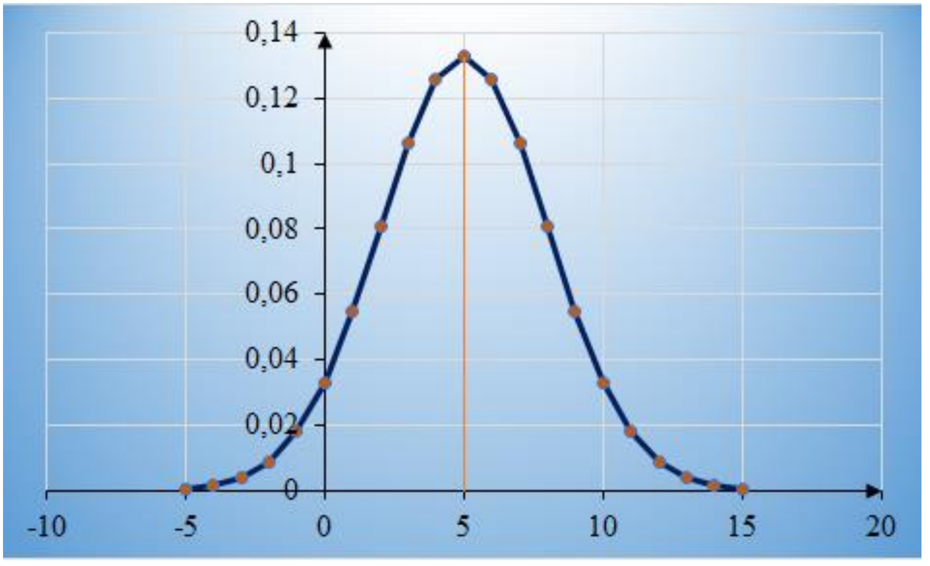
were a is a mathematical expectation of the random variable X, and σ is a standard deviation of X. In accordance with the condition of the task $M(x) = 5$ and $D(x) = 9$. Then we can write

$$a = M(x) = 5 \qquad \sigma = \sqrt{D(X)} = 3$$

So, the distribution density has the form

$$f(x) = \frac{1}{3\sqrt{2\pi}} e^{-\frac{(x-5)^2}{18}}$$

Let's plot the function $f(x)$



Task № 9

The mathematical expectation and standard deviation of the normally distributed random variable X are equal to 20 and 5, respectively. Find the probability that, as a result of the trial, X will take a value belonging to the interval (15, 25).

Solution.

The *probability* that X will take the value belonging to the interval (α, β) is determined by the formula

$$P(\alpha < x < \beta) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right),$$

where $\Phi(x)$ is the function of Laplace.

In our case $\alpha = 15,\quad \beta = 25,\quad a = 20,\quad \sigma = 5.$ Let's find the sought probability:

$$P(15 < x < 25) = \Phi\left(\frac{25 - 20}{5}\right) - \Phi\left(\frac{15 - 20}{5}\right) = \Phi(1) - \Phi(-1) =$$
$$= 0{,}3413 + 0{,}3413 = 0{,}6826$$

Task № 10

The time spent by the train at the sorting station is a random variable X, distributed according to the normal law with parameters $a = 3$ hours, $D(x) = 0{,}16$ hours. Find: the probability that the train will stay at the station: 1) from 2 to 3 hours; 2) less than 2,5 hours; 3) not less than 4 hours.

Solution.

The *probability* that X will take the value belonging to the interval (α, β) is determined by the formula

$$P(\alpha < x < \ \beta) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right),$$

where $\Phi(x)$ is the function of Laplace.

1) $\alpha = 2,\quad \beta = 3,\quad a = 3,\quad \sigma = \sqrt{D(x)} = \sqrt{0{,}16} = 0{,}4$. Let's find the sought probability:

$$P(2 < x < \ 3) = \Phi\left(\frac{3-3}{0{,}4}\right) - \Phi\left(\frac{2-3}{0{,}4}\right) = \Phi(0) - \Phi(-2{,}5)$$
$$= 0 + 0{,}4938 == 0{,}4938$$

2) $\alpha = 0;\quad \beta = 2{,}5;\quad a = 3;\quad \sigma = 0{,}4.$ Let's find the sought probability:

$$P(0 < x < \ 2{,}5) = \Phi\left(\frac{2{,}5-3}{0{,}4}\right) - \Phi\left(\frac{0-3}{0{,}4}\right) = \Phi(-1{,}25) - \Phi(-7{,}5) =$$
$$= -\Phi(1{,}25) + \Phi(7{,}5) = -0{,}3944 + 0{,}5 = 0{,}1056$$

3) $P(x \geq 4) = 1 - P(x < 4) = 1 - P(0 < x < \ 4) =$

$$= 1 - \Phi\left(\frac{4-3}{0{,}4}\right) + \Phi\left(\frac{0-3}{0{,}4}\right) = 1 - \Phi(2{,}5) - \Phi(7{,}5) =$$
$$= 1 - 0{,}4938 - 0{,}5 = 0{,}0062$$

PRACTICAL TASKS FOR INDEPENDENT WORK

1. A continuous random variable X is given by a distribution density function

$$f(x) = \begin{cases} 0, & if \quad x < 0 \\ \frac{1}{5}e^{-\frac{1}{5}x}, & if \ \ x \geq 0 \end{cases}$$

Find: 1) the distribution function; 2) the mathematical expectation M(x), dispersion D(x) and standard deviation $\sigma(x)$.

Answer: 1) $F(x) = \begin{cases} 0, & if \quad x < 0 \\ 1 - e^{-\frac{1}{5}x}, & if \ \ x \geq 0 \end{cases}$

2) $M(X) = 5$ $\quad D(X) = 25$ $\quad \sigma(X) = \sqrt{D(X)} = 5$

2. The mathematical expectation of a random variable X, which has an exponential distribution law, is equal to $1/7$. Write down: 1) the distribution function; 2) the distribution density function; 3) find the probability that the random variable X will belong to the interval $(-1; 1/7)$.

Answer:

1) $F(x) = \begin{cases} 0, & if \quad x < 0 \\ 1 - e^{-7x}, & if \ \ x \geq 0 \end{cases}$

2) $f(x) = \begin{cases} 0, & if \ \ x < 0 \\ 7e^{-7x}, & if \ \ x \geq 0 \end{cases}$

3) $P(-1 < X < 1/7) = 1 - 1/e$

3. A continuous random variable X is given by the distribution density function

$$f(x) = \begin{cases} 0, & if \quad x < 0 \\ Ae^{-\frac{2}{3}x}, & if \ \ x \geq 0 \end{cases}$$

Find: 1) the constant value A; 2) the mathematical expectation M(x), dispersion D(x) and standard deviation $\sigma(x)$.

Answer: 1) $\text{A} = 2/3$; *2)* $M(X) = 3/2$; *3)* $D(X) = 9/4$; *4)* $\sigma(x) = 3/2$

4. A continuous random variable X is given by a distribution density function

$$f(x) = \begin{cases} 0, & if \quad x < 0 \\ 0{,}04e^{-0{,}04x}, & if \ \ x \geq 0 \end{cases}$$

Find the probability that, as a result of the trial, X will belong to the interval (1, 2).

Answer: $P(1 < X < 2) = e^{-0{,}04} - e^{-0{,}08} \approx 0{,}038$

5. A continuous random variable X is given by a distribution function

$$F(x) = \begin{cases} 0, & if \quad x < 0 \\ 1 - e^{-0{,}06x}, & if \ \ x \geq 0 \end{cases}$$

Find the probability that, as a result of the trial, X will belong to the interval (2, 5).

Answer: $P(2 < X < 5) = e^{-0,12} - e^{-0,3} \approx 0,146$

6. A random variable X is distributed according to the normal law with the probability density

$$f(x) = \frac{1}{2\sqrt{2\pi}} e^{-\frac{(x-7)^2}{8}}$$

Find the mathematical expectation and standard deviation of X.
Answer: $M(x) = 7;\ \sigma = 2$

7. The mathematical expectation and standard deviation of the normally distributed random variable X are equal to 10 and 2 respectively. Find the probability that as a result of the trial X will take a value belonging to the interval (12, 14).
Answer: $P(12 < x < 14) = 0,1359$

SELF-TEST QUESTIONS

1. A random variable X is distributed according to the normal law with the probability density

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{(x-3)^2}{32}}$$

What is the standard deviation of X?
   a. 2
   b. 3
   c. 4

2. A continuous random variable X is uniformly distributed over the interval (1, 3). What is the mathematical expectation of X?
   a. 2
   b. 1
   c. 5

3. A continuous random variable X has an exponential distribution law with a parameter $\lambda = 2$. What is the value of the distribution function of X at the point x=1?
   a. $1 + e^{-3}$
   b. $1 - e^{-2}$
   c. $e^{-1} - 1$

4. A continuous random variable X has an exponential distribution law with a parameter $\lambda = 1/5$. What is the probability that, as a result of the trial, X falls into the interval (5, 10)?
   a. $e^{-1} - e^{-2}$
   b. $e^{-2} - e^{-1}$
   c. $e^{-1} - 1$

5. What is the dispersion of a discrete random variable X − the number of failures of an element of a certain device in 10 independent trials, if the probability of the element failure in each trial is equal to 0,9?
   a. 0,7
   b. 0,8
   c. 0,9

LITERATURE FOR SELF-STUDY

1. A.V. Tyurinand, A.Yu. Akhmerov Theory of probability and mathematical statistics: Textbook. Dusseldorf: LAP LAMBERT Academic Publishing GmbH & Co.KG., 2020. 148 p.
2. Prasanna Sahoo Probability and mathematical statistics: Textbook. USA: Department of Mathematics of the University of Louisville, 2013. 712 p.
3. J. K. Blitzstein, J. Hwang, Introduction to Probability Second Edition. Taylor & Francis Group, LLC, 2019. 636 p.

TOPIC 8. ELEMENTS OF MATHEMATICAL STATISTICS

PLAN

1. Statistical distribution of the sample.
2. Empirical distribution function.
3. Graphical representation of the sample. Plotting a polygon and a histogram of frequencies.

Key notions, definitions and categories to be studied: *The concept of the statistical distribution of the sample and the empirical distribution function. The notion that a polygon and a histogram are a graphic representation of a sample.*

PRACTICAL TASKS

EXAMPLES OF THE SOLVING OF PRACTICAL TASKS

Task № 1

In the study of the random variable X got the following values: -2, -4, -5, -4, -2, -4, 1, 0, 0, 1, 1, 2, 4, -4, -5, 4, 0, -2, -5, 0, 2, -5, 0, 1, 2. Find the size of sample. Make: a) the frequency distribution; b) the distribution of relative frequencies.

Solution.

a) The value «*-5*» occurs 4 times, the value «*-4*» occurs 4 times, …, the value «*4*» occurs 2 times. Let's make a table

x_i	-5	-4	-2	0	1	2	4
n_i	4	4	3	5	4	3	2

We write *the distribution of frequencies*. Let's write *the size of the sample*. We get $n = n_1 + n_2 + \cdots + n_k = 4 + 4 + 3 + 5 + 4 + +3 + 2 = 25$

b) Let's calculate *relative frequencies*. We can use the formula

$$w_i = {}^{n_i}/_{n} \quad i = 1, 2, 3, \dots k$$

$w_{1=}\ {}^{4}/_{25}$ $w_{2=}\ {}^{4}/_{25}$ $w_{3=}\ {}^{3}/_{25}$ $w_{4=}\ {}^{5}/_{25}$ $w_{5=}\ {}^{4}/_{25}$ $w_{6=}\ {}^{3}/_{25}$ $w_{7=}\ {}^{2}/_{25}$

Now we can write *the distribution of relative frequencies*

x_i	-5	-4	-2	0	1	2	4
w_i	4/25	4/25	3/25	5/25	4/25	3/25	2/25

Task № 2

In the study of the random variable X got the following values: -1, -7, -2, 1, 0, 1, -1, 1, 2, 6, -7, 6, 0, -2, -7, 0, 2, -7, 0, 1, -1, 2. Find the size of the sample. Make: a) the frequency distribution; b) the distribution of relative frequencies.

Solution.

a) The value «*-7*» occurs 4 times, the value «*-2*» occurs 2 times, …, the value «*6*» occurs 2 times. Let's make a table

x_i	-7	-2	-1	0	1	2	6
n_i	4	2	3	4	4	3	2

We write *the distribution of frequencies*. Let's write *the size of the sample*. We get $n = n_1 + n_2 + \cdots + n_k = 4 + 2 + 3 + 4 + 4 + +3 + 2 = 22$

b) Let's calculate *relative frequencies*. We can use the formula

$$w_i = {}^{n_i}/_{n} \quad i = 1, 2, 3, \dots k$$

$w_{1=}\ {}^{4}/_{22}$ $w_{2=}\ {}^{2}/_{22}$ $w_{3=}\ {}^{3}/_{22}$ $w_{4=}\ {}^{4}/_{22}$ $w_{5=}\ {}^{4}/_{22}$ $w_{6=}\ {}^{3}/_{22}$ $w_{7=}\ {}^{2}/_{22}$

Now we can write *the distribution of relative frequencies*

x_i	-7	-2	-1	0	1	2	6
w_i	4/22	2/22	3/22	4/22	4/22	3/22	2/22

Task № 3

During the experiment it was found that the measurement errors are: 0.7; 0; 0.7; 0.7; 0.3; 0.3; 0.2; 1; 0.2; 0.5; 0.5; 0.5. Find: 1) frequency distribution; 2) the empirical distribution function and plot it.

Solution.

1) The value «*0*» occurs once, the value «*0,2*» occurs 2 times, …, the value «*1*» occurs once. Let's make a table

x_i	0	0,2	0,3	0,5	0,7	1
n_i	1	2	2	3	3	1

We write *the distribution of frequencies*. Let's write *the sample size*. We get $n = n_1 + n_2 + \cdots + n_k = 1 + 2 + 2 + 3 + 3 + +1 = 12$

2) To find the empirical distribution function let's calculate relative frequencies

$$w_i = {}^{n_i}/_{n} \quad i = 1, 2, 3, \dots k$$

$$w_{1=}\, {}^{1}/_{12} \quad w_{2=}\, {}^{2}/_{12} \quad w_{3=}\, {}^{2}/_{12} \quad w_{4=}\, {}^{3}/_{12} \quad w_{5=}\, {}^{3}/_{12} \quad w_{6=}\, {}^{1}/_{12}$$

So, *the distribution of relative frequencies has the form*

x_i	0	0,2	0,3	0,5	0,7	1
w_i	1/12	2/12	2/12	3/12	3/12	1/12

Now we can find the empirical distribution function.
The smallest variant $x_1 = 0$, so

$$F^*(x) = 0, \quad if \quad x \le 0$$

$If \quad 0 < x \le 0{,}2$ then we can write

$$F^*(x) = w_1 = 1/12$$

$If \quad 0{,}2 < x \le 0{,}3$ then we can write

$$F^*(x) = w_1 + w_2 = \frac{1}{12} + \frac{2}{12} = \frac{3}{12}$$

$If \quad 0{,}3 < x \le 0{,}5$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 = \frac{1}{12} + \frac{2}{12} + \frac{2}{12} = \frac{5}{12}$$

$If \quad 0{,}5 < x \le 0{,}7$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 = \frac{1}{12} + \frac{2}{12} + \frac{2}{12} + \frac{3}{12} = \frac{8}{12}$$

$If \quad 0{,}7 < x \le 1$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 + w_5 = \frac{1}{12} + \frac{2}{12} + \frac{2}{12} + \frac{3}{12} = \frac{8}{12}$$

The largest variant $x_6 = 1$, therefore

$$F^*(x) = 1, \; if \quad x > 1$$

The empirical distribution function has the form:

$$F^*(x) = \begin{cases} 0 & x \le 0 \\ 1/12 & 0 < x \le 0{,}2 \\ 3/12 & 0{,}2 < x \le 0{,}3 \\ 5/12 & 0{,}3 < x \le 0{,}5 \\ 8/12 & 0{,}5 < x \le 0{,}7 \\ 11/12 & 0{,}7 < x \le 1 \\ 1 & x > 1 \end{cases}$$

Task № 4

We know the statistical distribution of the sample:

x_i	1	2	5	6	10	17	22
n_i	3	2	6	5	4	7	3

Find: 1) the distribution of relative frequencies; 2) the empirical distribution function; 3) plot a polygon of relative frequencies.

Solution.

Let's calculate *the size of the sample*. We get $n = n_1 + n_2 + \cdots + n_k = 3 + 2 + 6 + 5 + +4 + 7 + 3 = 30$

1) To find the empirical distribution function let's calculate relative frequencies

$$w_i = {}^{n_i}\!/_{n} \quad i = 1, 2, 3, \dots k$$

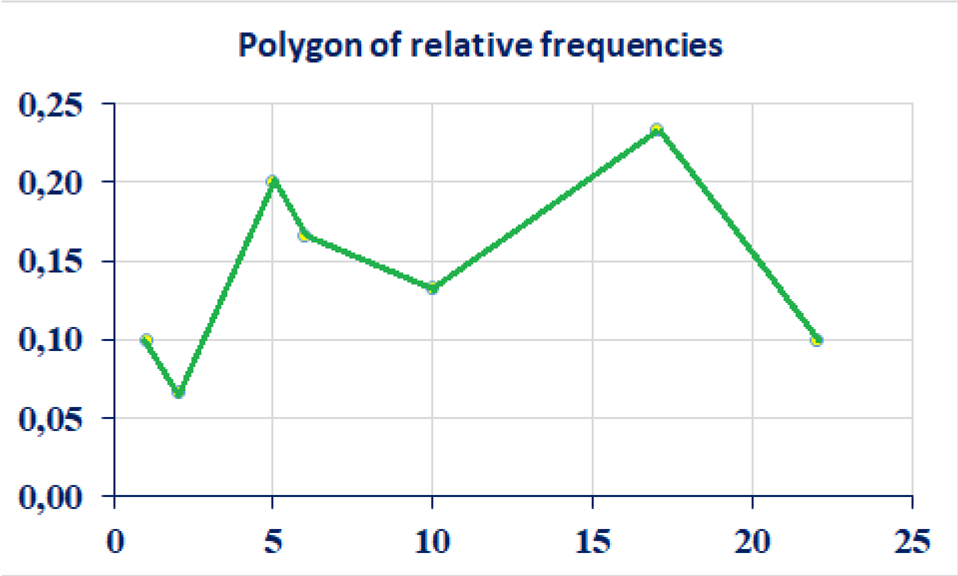
$w_1 = {}^{3}\!/_{30}$ $w_2 = {}^{2}\!/_{30}$ $w_3 = {}^{6}\!/_{30}$ $w_4 = {}^{5}\!/_{30}$ $w_5 = {}^{4}\!/_{30}$ $w_6 = {}^{7}\!/_{30}$

$w_7 = {}^{3}\!/_{30}$

So, *the distribution of relative frequencies has the form*

x_i	1	2	5	6	10	17	22
w_i	3/30	2/30	6/30	5/30	4/30	7/30	3/30

Let's plot a polygon of relative frequencies.



Now we can find the empirical distribution function.
The smallest variant $x_1 = 1$, so

$$F^*(x) = 0, \quad if \quad x \le 1$$

$If \quad 1 < x \le 2$ then we can write

$$F^*(x) = w_1 = 3/30$$

$If \quad 2 < x \le 5$ then we can write

$$F^*(x) = w_1 + w_2 = \frac{3}{30} + \frac{2}{30} = \frac{5}{30}$$

$If \quad 5 < x \le 6$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 = \frac{3}{30} + \frac{2}{30} + \frac{6}{30} = \frac{11}{30}$$

$If \quad 6 < x \le 10$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 = \frac{3}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} = \frac{16}{30}$$

$If \quad 10 < x \le 17$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 + w_5 = \frac{3}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} + \frac{4}{30} = \frac{20}{30}$$

$If \quad 17 < x \le 22$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 + w_5 + w_6 = \frac{3}{30} + \frac{2}{30} + \frac{6}{30} + \frac{5}{30} + \frac{4}{30} + \frac{7}{30} = \frac{27}{30}$$

The largest variant $x_6 = 22$, therefore

$$F^*(x) = 1, \ if \quad x > 22$$

The empirical distribution function has the form:

$$F^*(x) = \begin{cases} 0 & x \le 1 \\ 3/30 & 1 < x \le 2 \\ 5/30 & 2 < x \le 5 \\ 11/30 & 5 < x \le 6 \\ 16/30 & 6 < x \le 10 \\ 20/30 & 10 < x \le 17 \\ 27/30 & 17 < x \le 22 \\ 1 & x > 22 \end{cases}$$

Task № 5

We know the statistical distribution of the sample:

x_i	1	2	3	6	7	9	14
n_i	2	3	5	9	11	8	7

Find: 1) ordinates of the frequency polygon for the variants: x =3; x=14; 2) ordinates of the polygon of the relative frequencies for the variants: x=2; x=7; 3) the value of the empirical distribution function $F^*(x)$ at intervals: (2,3], (7,9]; 4) write the empirical distribution function $F^*(x)$.

Solution.

1) Let's find the ordinates of the frequency polygon for the given variants:
for x= 3 the ordinate of the frequency polygon is *n* =5;
for x = 14 the ordinate of the frequency polygon is *n* =7.

2) *The size of the sample* is $n = n_1 + n_2 + \cdots + n_k = 2 + 3 + 5 + 9 + 11 + 8 + +7 = 45$

Let's calculate *relative frequencies*

$$w_i = {n_i}/{n} \quad i = 1, 2, 3, \dots k$$

$w_1 = {2}/{45}$ $w_2 = {3}/{45}$ $w_3 = {5}/{45}$ $w_4 = {9}/{45}$ $w_5 = {11}/{45}$ $w_6 = {8}/{45}$ $w_7 = {7}/{45}$

So, *the distribution of relative frequencies has the form*

x_i	1	2	3	6	7	9	14
w_i	2/45	3/45	5/45	9/45	11/45	8/45	7/45

So, relative frequencies for the given variants:
for x=2 relative frequency is $w_2 = 3/45$
for x=7 relative frequency is $w_{10} = 11/45$

3) Let's find the empirical *distribution function* $F^*(x)$.

$If \;\; x \le 1$ then $F^*(x) = 0$

$If \;\; 1 < x \le 2$ then we can write

$$F^*(x) = w_1 = 2/45$$

$If \;\; 2 < x \le 3$ then we can write

$$F^*(x) = w_1 + w_2 = \frac{2}{45} + \frac{3}{45} = \frac{5}{45}$$

$If \;\; 3 < x \le 6$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 = \frac{2}{45} + \frac{3}{45} + \frac{5}{45} = \frac{10}{45}$$

$If \;\; 6 < x \le 7$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 = \frac{2}{45} + \frac{3}{45} + \frac{5}{45} + \frac{9}{45} = \frac{19}{45}$$

$If \;\; 7 < x \le 9$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 + w_5 = \frac{2}{45} + \frac{3}{45} + \frac{5}{45} + \frac{9}{45} + \frac{11}{45} = \frac{30}{45}$$

$If \;\; 9 < x \le 14$ then we can write

$$F^*(x) = w_1 + w_2 + w_3 + w_4 + w_5 + w_6$$
$$= \frac{2}{45} + \frac{3}{45} + \frac{5}{45} + \frac{9}{45} + \frac{11}{45} + \frac{8}{45} == \frac{38}{45}$$

$If \;\; x > 14$ then $F^*(x) = 1$

The empirical distribution function has the form:

$$F^*(x) = \begin{cases} 0 & x \le 1 \\ 2/45 & 1 < x \le 2 \\ 5/45 & 2 < x \le 3 \\ 10/45 & 3 < x \le 6 \\ 19/45 & 6 < x \le 7 \\ 30/45 & 7 < x \le 9 \\ 38/45 & 9 < x \le 14 \\ 1 & x > 14 \end{cases}$$

So, sought values of the empirical distribution function are

$$F^*(x) = \frac{5}{45} \quad if \;\; 2 < x \le 3$$

$$F^*(x) = \frac{30}{45} \quad if \; 7 < x \le 9$$

Task № 6

We know the statistical distribution of the sample

x_i	0	1	1,9	2,1	3,3	3,8	4,1	4,4	5,4	6,0
n_i	1	2	4	4	9	12	8	6	3	1

Calculate:

1) the height of the k-th rectangle of the frequency histogram, if k = 3, h = 2;

2) the height of the k-th rectangle of the histogram of relative frequencies, if k = 2, h = 2;

Solution.

Let's find the sample size: n= 1+2+4+4+9+12+8+6+3+1=50.

Find the number of partial intervals. Let's define

$$x_{min} = 0 \quad x_{max} = 6$$

Then the number of partial intervals will be equal to

$$k = \frac{x_{max} - x_{min}}{h} = \frac{6-0}{2} = 3$$

For each interval, calculate the frequencies n_i, the relative frequencies w_i and the relations ${n_i}/{h}$, ${w_i}/{h}$. The data are recorded into the table:

	1	2	3
Interval	0;2	2;4	4;6
n_i	7	25	18
w_i	7/50	25/50	18/50
n_i/h	7/2	25/2	18/2
w_i/h	0,07	0,25	0,18

The height of the 3-rd rectangle of the frequency histogram is calculated by the formula:

$$\frac{n_3}{h}=\frac{18}{2}=9$$

The height of the 2-nd rectangle of the histogram of relative frequencies is

$$\frac{w_2}{h}=0{,}25$$

Task № 7

In the study of the random variable X got the following values: 0,6; 0,8; 0,61; 0,72; 0,69; 0,68; 0,68; 0,72; 0,74; 0,68. Make: 1) an interval series with a step h = 0,05; 2) plot a histogram of frequencies; 3) plot a histogram of relative frequencies.

Solution.

Let's find the size of the sample: n=10. Find the number of partial intervals. We will define

$$x_{min}=0{,}6$$
$$x_{max}=0{,}8$$

Then the number of partial intervals will be equal to

$$k=\frac{x_{max}-x_{min}}{h}=\frac{0{,}8-0{,}6}{0{,}05}=4$$

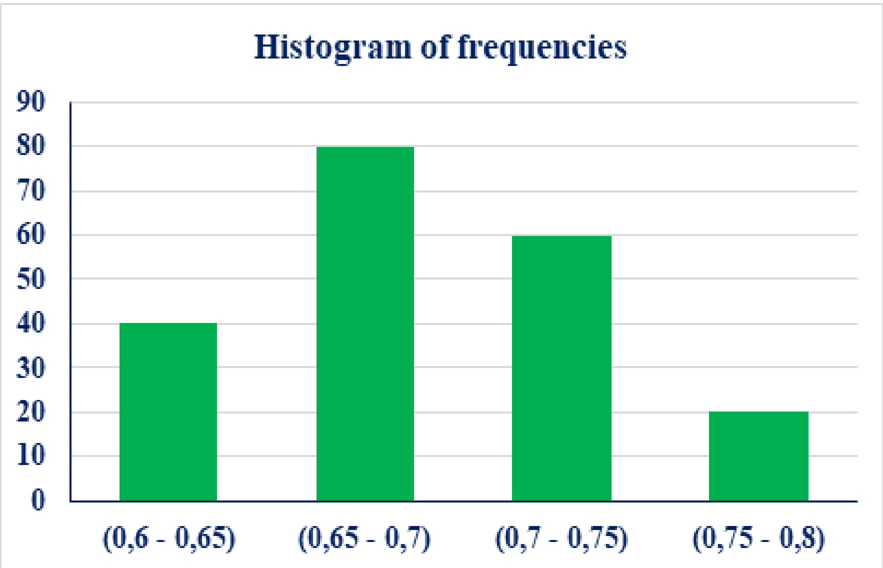
Let's make an interval series with a step h = 0,05. We get the next result

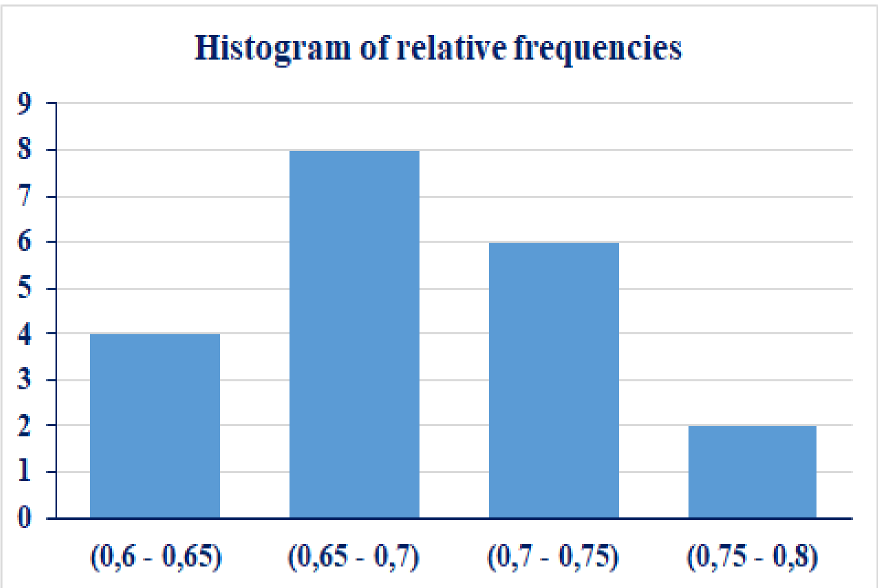
x_i -	0,6-0,65	0,65-0,7	0,7-0,75	0,75-0,8
n_i	2	4	3	1

For each interval, calculate the relative frequencies w_i and the density n_i/h, w_i/h. The data are recorded into the table:

	1	2	3	4
Interval	0,6-0,65	0,65-0,7	0,7-0,75	0,75-0,8
n_i	2	4	3	1
w_i	2/10	4/10	3/10	1/10
n_i/h	40	80	60	20
w_i/h	4	8	6	2

Let's plot a histogram of frequencies and a histogram of relative frequencies.





PRACTICAL TASKS FOR INDEPENDENT WORK

Task № 1

We know the statistical distribution of the sample:

x_i	1	3	5	7	9
n_i	2	4	6	8	10

Find: 1) the ordinates of the frequency polygon for the variants: x = 3; x = 7; 2) the ordinates of the polygon of the relative frequencies for the variants: x = 5; x = 9; 3) the value of the empirical distribution function $F^*(x)$ at interval: (1,7].

Task № 2

Plot a histogram of frequencies by a given sample distribution

	1	2	3	4	5
Interval	2- 7	7- 12	12- 17	17-22	22-27
n_i	5	10	25	6	4

Task № 3

Plot a histogram of relative frequencies by a given sample distribution

	1	2	3	4
Interval	2- 4	4- 6	6- 8	8-10
n_i	6	10	4	5

SELF-TEST QUESTIONS

1. In the study of the random variable X, the following values were obtained: -5; -4; -2; 0; 1; 1; 2; 2; 3; 4. Find the size of the sample.
   a. 8
   b. 5
   c. 10

2. In the study of the random variable X, the following values were obtained: -0,3; -0,4; -0,4; 0,1; 0; -0,2; -0,1; -0,2; -0,2; -0,3. Find the frequency of the variant the value of which is equal to -0,2.
   a. 3
   b. 5
   c. 4

3. In the study of the random variable X, the following values were obtained: 0; 1; 3; 1; 2; 3; 4; 3; 4; 3. Find the relative frequency of the variant the value of which is equal to 3.
   a. 0,2
   b. 0,4
   c. 0,3

4. In the study of the random variable X, the following values were obtained: 1, 2, 1, 3, 3, 2, 6, 2, 3, 6. Find the ordinate of the relative frequency polygon for variant the value of which is equal to 6
   a. 0,5
   b. 0,4
   c. 0,2

5. In the study of the random variable X, the following values were obtained: -0,3; -0,4; -0,4; 0,1; 0; -0,2; -0,1; -0,2; -0,2; -0,3. Find the value of the empirical distribution function on the interval (-0,4; -0,3]
   a. 0,2
   b. 0,1
   c. 0,3

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ANSWERS TO SELF-TEST QUESTIONS

	CHAPTER							
	1	2	3	4	5	6	7	8
1	a	a	c	a	b	a	c	c
2	c	b	b	b	c	b	a	a
3	b	a	b	a	a	a	b	b
4	a	b	c	b	b	b	a	c
5	a	b	b	c	b	a	c	a

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ANNEX

BASIC CALCULATION FORMULAS

TOPIC 1. BASIC CONCEPTS OF THE THEORY OF PROBABILITY. ELEMENTS OF COMBINATORICS

FORMULA	DESCRIPTION
$P_n = n!$	Formula of permutations
$P_n(n_1, n_2, \dots n_k) = \frac{n!}{n_1! \cdot n_2! \cdot \dots \cdot n_k!}$	Formula of permutations with repetitions
$A_n^m = \frac{n!}{(n-m)!}$	Placements formula
$C_n^m = \frac{n!}{m!\,(n-m)!}$	Combinations formula
$P(A) = \frac{m}{n}$	Classical definition of probability

TOPIC 2. THEOREMS OF ADDITION AND MULTIPLICATION OF PROBABILITIES

FORMULA	DESCRIPTION
$P(A+B) = P(A) + P(B) - P(AB)$	The theorem of addition of probabilities (compatible events)
$P(A+B) = P(A) + P(B)$	The theorem of addition of probabilities (incompatible events)
$P(A) + P(\bar{A}) = 1$	Equality defining the dependency between opposite events
$P(A \cdot B) = P(A) \cdot P_A(B) = P(B) \cdot P_B(A)$	The theorem of multiplication of probabilities (dependent events)
$P(A \cdot B) = P(A) \cdot P(B)$	The theorem of multiplication of probabilities (independent events)

TOPIC 3-4. FORMULA OF COMPLETE PROBABILITY. BAYES' FORMULA. BERNOULLI'S FORMULA. LOCAL AND INTEGRAL THEOREMS OF LAPLACE

FORMULA	DESCRIPTION
$P(A)=P(B_1)\cdot P_{B_1}(A)+P(B_2)\cdot P_{B_2}(A)+\cdots+P(B_n)\cdot P_{B_n}(A)$	Formula of complete probability
$P_A(B_i)=\dfrac{P(B_i)\cdot P_{B_i}(A)}{P(A)}$, $(i=1,2,\dots,n)$ where $P(A)=P(B_1)\cdot P_{B_1}(A)+P(B_2)\cdot P_{B_2}(A)+\cdots+P(B_n)\cdot P_{B_n}(A)$.	Bayes' formula
$P_n(m)=C_n^m\cdot p^m\cdot q^{n-m}$, $\quad q=1-p$	Bernoulli's formula
$P_n(m)\approx\dfrac{\lambda^m}{m!}e^{-\lambda}$, $\quad \lambda=np$.	Poisson's formula
$P_n(m)\approx\dfrac{1}{\sqrt{npq}}\varphi(x_m)$, $\quad x_m=\dfrac{m-np}{\sqrt{npq}}$	Local theorem of Laplace
$P(m_1,m_2)=\Phi(\ddot{x})-\Phi(\dot{x})$, where $\Phi(x)=\frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{z^2}{2}}dz$ – *Laplace function*, $\dot{x}=\dfrac{m_1-np}{\sqrt{npq}}$ $\ddot{x}=\dfrac{m_2-np}{\sqrt{npq}}$	Integral Laplace theorem
$\Phi(-x)=-\Phi(x)$	Laplace function is odd

TOPIC 5. DISCRETE AND CONTINUOUS RANDOM VARIABLES. INTEGRAL DISTRIBUTION FUNCTION. DIFFERENTIAL DISTRIBUTION FUNCTION

FORMULA	DESCRIPTION
$F(x)=P(X < x)$	The integral distribution function of a random variable X
$P(a < X < b) = F(b) - F(a)$	The probability that a random variable X will take a value from the segment [a, b]
$F(x) = 0 \ when \ \ x \leq a;$ $F(x) = 1 \ when \ x \geq b.$ $0 \leq F(x) \leq 1$	Properties of the integral distribution function
$f(x) = F'(x)$	The differential distribution function of a continuous random variable X
$P(a < x < b) = \int_a^b f(x)dx$	The probability that a continuous random variable will take a value belonging to the interval (a, b)
$\int_a^b f(x)dx = 1$ $f(x) \geq 0$	Properties of the differential distribution function
$F(x) = \int_{-\infty}^{x} f(x)dx$	Relation between integral and differential functions

TOPIC 6. NUMERICAL CHARACTERISTICS OF RANDOM VARIABLES

FORMULA	DESCRIPTION
$M(X) = x_1 \cdot p_1 + x_2 \cdot p_2 + \cdots + x_n \cdot p_n$	The mathematical expectation of a discrete random variable
$\mathrm{M(C) = C}, \qquad \mathrm{C} - const$ $\mathrm{M(CX) = CM(X)}, \qquad \mathrm{C} - const$ $\mathrm{M}(\mathrm{X}_1 \cdot \mathrm{X}_2 \cdot \ldots \cdot \mathrm{X}_n) =$ $= \mathrm{M}(\mathrm{X}_1) \cdot \mathrm{M}(\mathrm{X}_2) \cdot \ldots \cdot \mathrm{M}(\mathrm{X}_n)$ $\mathrm{M}(\mathrm{X}_1 + \mathrm{X}_2 + \cdots + \mathrm{X}_n)$ $= \mathrm{M}(\mathrm{X}_1) + \mathrm{M}(\mathrm{X}_2) + \cdots + \mathrm{M}(\mathrm{X}_n)$	Properties of the mathematical expectation

$D(X)=M(X^2)-[M(X)]^2$	The dispersion of a discrete random variable
$D(\mathrm{CX})=C^2D(\mathrm{X}), \quad \mathrm{C}-const$ $D(\mathrm{X}_1+\mathrm{X}_2+\dots+\mathrm{X}_n)$ $=D(\mathrm{X}_1)+D(\mathrm{X}_2)+\dots+D(\mathrm{X}_n)$	Properties of the dispersion
$M(X)=\int_a^b xf(x)\,dx$	The mathematical expectation of a continuous random variable
$D(X)=\int_a^b x^2f(x)\,dx-\ [M(X)]^2$	The dispersion of a continuous random variable
$\sigma(X)=\sqrt{D(X)}$	The standard deviation

TOPIC 7. DISTRIBUTION LAWS OF RANDOM VARIABLES

FORMULA	DESCRIPTION
$P_n(m)=C_n^m\cdot p^m\cdot q^{n-m}$, $q=1-p$ $M(X)=np,\ D(X)=npq$, $\sigma(X)=\sqrt{D(X)}=\sqrt{npq}$	The binomial law of the distribution
$P_n(m)\approx\frac{\lambda^m}{m!}e^{-\lambda}, \quad \lambda=np$ $M(X)=\ \lambda,\ D(X)=\lambda,\ \ \sigma(X)=\sqrt{D(X)}=\sqrt{\lambda}$	The Poisson’s law of the distribution
$f(x)=\begin{cases}\frac{1}{b-a}, & if\ \ x\epsilon(a,b)\\ 0, & if\ \ x\notin(a,b)\end{cases}$ $M(X)=\frac{a+b}{2}$ $D(X)=\frac{(b-a)^2}{12}$ $\sigma(X)=\sqrt{D(X)}=\frac{b-a}{2\sqrt{3}}$	The uniform law of the distribution on the interval (a, b)

$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}$, where a is the mathematical expectation and σ is the standard deviation of X.	The normal law of the distribution
$P(\alpha < x < \beta) = \Phi\left(\frac{\beta - a}{\sigma}\right) - \Phi\left(\frac{\alpha - a}{\sigma}\right)$, $\Phi(x)$ is the Laplace function	The probability that X (a normally distributed random variable) will take the value belonging to the interval (α, β)
$f(x) = \begin{cases} 0, & if \ x < 0 \\ \lambda e^{-\lambda x}, & if \ x \geq 0 \end{cases}$ $M(X) = \frac{1}{\lambda}$ $D(X) = \frac{1}{\lambda^2}, \ \sigma(X) = \sqrt{D(X)} = \frac{1}{\lambda}$	The exponential law of the distribution
$P(a < x < b) = e^{-\lambda a} - e^{-\lambda b}$	The probability that X (a random variable distributed by the exponential law) belongs to the interval (a, b)

TOPIC 8. ELEMENTS OF MATHEMATICAL STATISTICS

FORMULA	DESCRIPTION
$n_1 + n_2 + \dots + n_k = n$	The sum of all frequencies is equal to the sample size n
$w_1 + w_2 + \dots + w_k = 1$, $w_i = {}^{n_i}/_{n}, \quad i = \overline{1,k}$	The sum of the relative frequencies is equal to one
$F^*(x) = \frac{n_x}{n}$, where n_x –is the number of variants less than x; n –is the sample size.	The empirical distribution function

$F^*(x) \in [0,1]$; $F^*(x) -$ is non-decreasing function; If $x_1 -$ is the smallest variant, and $x_k -$ is the largest variant, then $F^*(x) = 0$, if $x \le x_1$ and $F^*(x) = 1$, if $x > x_k$.	Properties of the empirical distribution function
A frequency polygon is a polyline which segments connect points (x_1, n_1), (x_2, n_2),…, (x_k, n_k), where $x_i -$ variants of the sample, $n_i -$corresponding frequencies.	Definition of the frequency polygon
A relative frequency polygon is a polyline which segments connect points (x_1, w_1), (x_2, w_2),…, (x_k, w_k), where $x_i -$ variants of the sample, $w_i -$corresponding relative frequencies.	Definition of the relative frequency polygon
The histogram of frequencies is called a stepped figure, which is composed of rectangles, the bases of which are partial intervals of length h, and the heights are equal to the ratio ${n_i}/{h}$	Definition of the histogram of frequencies

Навчальне видання

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ТЕОРІЯ ЙМОВІРНОСТЕЙ ТА МАТЕМАТИЧНА СТАТИСТИКА: ПРИКЛАДИ ТА ПРОБЛЕМИ

Навчальний посібник для здобувачів економічних спеціальностей

Англійською мовою

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